AN ANALYSIS OF DUAL FORMULATIONS FOR THE FINITE ELEMENT SOLUTION OF TWO-BODY CONTACT PROBLEMS

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Table of Contents

1. Introduction
2. Background on Contact Mechanics
3. A Discrete Babuška-Brezzi Condition for Two-body Contact
4. Convergence and Error Analysis
5. A Review of Formulations for the Signorini problem
6. Two-body Contact Formulations
7. An Alternative Pressure Interpolation for Two-body Contact
8. A Numerical Test
9. Conclusions

References

Appendix A: Fractional Sobolev Spaces for Two-body Contact
Appendix B: Lemmas and Theorems on the $\| \cdot \|_{\Gamma/2}$ norm
Appendix C: Inverse Assumption for Constant-size Discretizations of 1-D Domains using Fractional Norms
Appendix D: Babuška-Brezzi Condition for Node-on-Surface Methods
Appendix E: Some Results on the Babuška-Brezzi Condition for Two-body Contact
Appendix F: Babuška-Brezzi Condition for an Alternative Pressure Interpolation

Abstract

This article examines the convergence properties of dual finite element formulations of the two-dimensional frictionless two-body contact problem under the assumption of infinitesimal kinematics. The centerpiece of the proposed analysis is the well-known Babuška-Brezzi condition, suitably adapted to the present problem. It is demonstrated for certain canonical geometries that several widely used methods that employ pressure or force interpolations derived from the discretizations of both surfaces violate the Babuška-Brezzi condition, thus producing increasingly oscillatory solutions under mesh refinement. Alternative algorithms are proposed that circumvent this difficulty and are shown to yield convergent solutions.

Keywords: Finite element method, two-body contact, dual formulation, node-on-surface, Babuška-Brezzi condition, convergence.

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1 Introduction

Dual formulations are commonly employed in the numerical solution of problems of contact between deformable solids. In such formulations, Lagrange multipliers are introduced in order to enforce the unilateral constraint of impenetrability of matter over the full (i.e., unconstrained) space of admissible solutions. As in other classes of problems featuring kinematic constraints (e.g., incompressibility in elastic or linear viscous media, vanishing of the transverse shear in Reissner plates at the thin-plate limit), dual formulations of the two-body contact problem require careful selection of the interpolation of both primal and dual variables, as well as the integration rule for the associated weak forms. Otherwise, the resulting finite element solutions may exhibit instabilities analogous in nature to the “checkerboard mode” in incompressible elasticity [1, page 212] or may be subject to the imposition of artificial kinematic constraints, a condition referred to in [2] as “surface locking”. In both cases, the resulting finite element solutions may not converge under mesh refinement.

To date, a general convergence analysis of dual finite element methods for the two-body contact problem is not available. However, basic ingredients of such an analysis have been put in place by Oden, Kikuchi and coworkers [3, 4, 5, 6] in connection with reduced integration penalty methods for the two-dimensional Signorini problem, i.e., the problem of contact between a linearly elastic body and a stationary rigid foundation. These authors formulated an inequality, based upon earlier work by Ladyzhenskaia [7], Babuška [8] and Brezzi [9], whose satisfaction was shown to yield convergent approximations for reduced integration penalty methods. No firm conclusions were drawn regarding the convergence of formulations which fail the aforementioned condition, although it was noted that numerical experimentation suggested they are deficient, see [4]. The present article proposes a Ladyzhenskaia-Babuška-Brezzi (LBB) condition for the two-body contact problem under certain simplifying assumptions deemed necessary to render the analysis tractable. In addition, it includes convergence theorems based on the satisfaction of the proposed Babuška-Brezzi condition and assumed approximations for the primal and dual fields. Furthermore, the article assesses the convergence properties of a wide class of dual finite element approximations for certain canonical Signorini and two-body contact problems. The practical outcome of this analysis is the rigorous evaluation of node-on-surface methods commonly used in commercial codes, as well as a series of dual methods based on smooth interpolations.
The organization of the article is as follows: essential information on contact mechanics and related mathematical background is contained in Section 2. The formulation of continuous and discrete Babuška-Brezzi conditions for the two-body contact problem are presented in Section 3, and are followed by associated convergence results in Section 4. The special case of the Signorini problem is discussed in Section 5, while Section 6 is devoted to the analysis of classical finite element methods for the two-body problem. This is followed by a similar analysis of alternative dual formulations in Section 7. A simple numerical test is included in Section 8 to assess the validity of the theoretical findings. Concluding remarks are offered in Section 9.

2 Background on Contact Mechanics

Consider the problem of contact between two deformable bodies \( B^\alpha \) and \( B^\beta \). Throughout the article, the superscript \( \alpha (= 1 \text{ or } 2) \) specifies one of the two bodies, while the superscript \( \beta = \text{mod} (\alpha, 2) + 1 \) specifies the other. Let \( B^\alpha \) occupy at some reference time \( t = 0 \) the open region \( \Omega^\alpha \in \mathcal{E}^d \), where \( \mathcal{E}^d \) is the Euclidean point space of dimension \( d \) (= 2 or 3). Correspondingly, let the same body occupy the open region \( \Omega^\alpha_t \in \mathcal{E}^d \) at time \( t \). A material point is identified at time \( t = 0 \) by its position \( X^\alpha \). The placement of the body is an invertible map \( \chi^\alpha_t(X^\alpha) \) associating material points \( X^\alpha \) at time \( t = 0 \) to their place \( x \in \Omega^\alpha_t \) at time \( t \). In addition, the displacement vector at time \( t \) is given by \( u^\alpha = x^\alpha - X^\alpha \).

The (orientable) boundaries of \( \Omega^\alpha \) and \( \Omega^\alpha_t \) are denoted by \( \partial \Omega^\alpha \) and \( \partial \Omega^\alpha_t \) and possess outward unit normals \( N^\alpha \) and \( n^\alpha \), respectively. Let the boundary \( \partial \Omega^\alpha_t \) be partitioned into three (not necessarily disjoint) regions as

\[
\partial \Omega^\alpha_t = \Gamma^\alpha_{u,t} \cup \Gamma^\alpha_{q,t} \cup \Gamma^\alpha_{c,t}.
\]  

(1)

Dirichlet and Neumann boundary conditions apply on \( \Gamma^\alpha_{u,t} \) and \( \Gamma^\alpha_{q,t} \), respectively, while \( \Gamma^\alpha_{c,t} \) is the region of contact with the other body. An analogous decomposition of \( \partial \Omega^\alpha \) into boundary regions \( \Gamma^\alpha_u \), \( \Gamma^\alpha_q \), and \( \Gamma^\alpha_c \) is naturally effected at time \( t = 0 \). In order to formally define the contact regions, first recall the principle of impenetrability of matter [10, page 244], which requires that

\[
\Omega^\alpha_t \cap \Omega^\beta_t = \emptyset,
\]  

(2)

for all \( t \). A direct link between the motion and the impenetrability constraint may be established by way of a “gap” function \( g^\alpha_t \) defined on \( \partial \Omega^\alpha_t \). A general set-theoretic definition
of such a gap function and its physical meaning are as follows:

\[ g_t^\alpha(x^\alpha) \begin{cases} 
> 0 & \iff x^\alpha \notin \Omega_t^\beta \iff \text{no contact} \\
= 0 & \iff x^\alpha \in \partial \Omega_t^\beta \iff \text{contact} \\
< 0 & \iff x^\alpha \in \Omega_t^\beta \iff \text{penetration} 
\end{cases} \]  

(3)

Hence, the contact region \( \Gamma_{c,t}^\alpha \) is comprised of all points on \( \partial \Omega_t^\alpha \) for which \( g_t^\alpha(x^\alpha) = 0 \).

For future use, boundary regions \( \Gamma_p^\alpha \) and \( \Gamma_{p,t}^\alpha \) of potential contact are also defined respectively at time \( t = 0 \) and \( t \) as consisting of all material points that may come to contact at any time during the deformation process. It is assumed that all points of \( \Gamma_p^\alpha \) and \( \Gamma_{p,t}^\alpha \) not in contact are only subject to homogeneous Neumann boundary conditions. It is further assumed that the boundaries \( \Gamma_p^\alpha \) are smooth, in that they possess unique outward unit normals. The definition of the actual and potential contact regions implies that, at time \( t = 0 \),

\[ \Gamma_c^\alpha = \Gamma_c^\beta = \Gamma_c, \quad \Gamma_c^\alpha \subseteq \Gamma_p^\alpha \subseteq \partial \Omega^\alpha, \]  

(4)

and, similarly, at time \( t \),

\[ \Gamma_{c,t}^\alpha = \Gamma_{c,t}^\beta = \Gamma_{c,t}, \quad \Gamma_{c,t}^\alpha \subseteq \Gamma_{p,t}^\alpha \subseteq \partial \Omega_t^\alpha. \]  

(5)

In equations (4) and (5), \( \Gamma_c \) and \( \Gamma_{c,t} \) denote the (common) contact boundary of the two bodies at times \( t = 0 \) and \( t \), respectively.

Particular choices for a gap function that conform to the general conditions in (3) can be found in [11, 12]. Here, the gap function is defined as

\[ g_t^\alpha(x^\alpha) = (\dot{x}^\beta - x^\alpha) \cdot n^\alpha, \]  

(6)

where the point \( \dot{x}^\beta = \pi_t^\alpha(x^\alpha) \) is the closest projection of \( x^\alpha \in \Gamma_{p,t}^\alpha \) onto \( \partial \Omega_t^\beta \) along the normal \( n^\alpha \) at time \( t \), and \( \pi_t^\alpha : \Gamma_{p,t}^\alpha \mapsto \Gamma_{p,t}^\beta \) is the projection function. In case the surfaces \( \Gamma_{p,t}^\alpha \) do not possess a unique normal, some particular choice of a normal needs to be made. However, this will problem will not be encountered in the ensuing analysis. More details about the complete definition of the gap function may be found in [12].

At this stage, certain simplifying assumptions are introduced in order to render the ensuing analysis of the two-body contact problem mathematically tractable: First, the deformations are taken to be elastic and infinitesimal relative to the configuration at time \( t = 0 \). Also, the loading conditions are considered quasi-static with \( \Gamma, \Gamma_{u,t} \) non-empty, and such that all rigid-body modes are suppressed. Also, contact is assumed to occur in the absence of friction or adhesion between the two surfaces. Under these conditions, the equations that
govern the motion of the two contacting bodies at time $t$ can be resolved on the geometry of the reference configuration and take the form

$$\text{div}^\alpha \sigma^\alpha + \rho_0^\alpha b^\alpha = 0 \quad \text{in } \Omega^\alpha ,$$

$$\sigma^\alpha \mathbf{N}^\alpha = \mathbf{t}^\alpha \quad \text{on } \Gamma_q^\alpha ,$$

$$u^\alpha = 0 \quad \text{on } \Gamma_u^\alpha ,$$

$$\sigma^\alpha \mathbf{N}^\alpha \cdot \mathbf{N}^\alpha = -p^\alpha \quad \text{on } \Gamma_c^\alpha ,$$

$$p^\alpha = p^\beta \quad \text{on } \Gamma_c^\alpha ,$$

$$p^\alpha \geq 0 \quad \text{on } \Gamma_p^\alpha ,$$

$$g^\alpha \geq 0 \quad \text{on } \Gamma_p^\alpha ,$$

$$p^\alpha g^\alpha = 0 \quad \text{on } \Gamma_p^\alpha .$$

Equation (7)

In the above, $\rho_0^\alpha$ is the referential mass density, $b^\alpha$ the body force per unit mass, $\sigma^\alpha = C^\alpha \frac{1}{2}(\nabla u^\alpha + \nabla^T u^\alpha)$ the (symmetric) stress tensor of the infinitesimal theory, $C^\alpha$ the elasticity modulus, $p^\alpha$ the contact pressure, and $\mathbf{t}^\alpha$ the prescribed traction. Also, $g^\alpha$ is the linearized counterpart of the gap function (6) relative to the reference configuration, written as

$$g^\alpha (X^\alpha; u^\alpha, u^\beta) = [(\mathbf{\dot{X}}^\beta + \mathbf{u}^\beta (\mathbf{\dot{X}}^\beta)) - (X^\alpha + u^\alpha)] \cdot \mathbf{N}^\alpha ,$$

Equation (8)

where $\mathbf{\dot{X}}^\beta = \pi^\alpha (X^\alpha)$ and $\pi^\alpha : \Gamma_p^\alpha \mapsto \Gamma_p^\beta$ is the referential projection function. Also, note that the Dirichlet boundary conditions in (7) are taken to be homogeneous without any substantial loss of generality.

Another simplification introduced here is that the normal projection $\pi^\alpha$ maps the same pairs of material points from one potential contact surface to the other, namely that

$$\pi^\alpha (X^\alpha) = (\pi^\beta)^{-1} (X^\alpha) .$$

Equation (9)

This simplification implies that

$$\mathbf{N}^\alpha (X^\alpha) = -\mathbf{N}^\beta (\pi^\alpha (X^\alpha)) ,$$

Equation (10)

so that, taking into account (8),

$$g^\alpha (X^\alpha; u^\alpha, u^\beta) = g^\beta (\pi^\alpha (X^\alpha); u^\beta, u^\alpha) .$$

Equation (11)

Furthermore, equations (7) necessitate that

$$p^\alpha (X^\alpha) = p^\beta (\pi^\alpha (X^\alpha))$$

Equation (12)
throughout $\Gamma_p^\alpha$. Consequently, integrals over the potential contact boundary of one body can be readily written on the respective boundary of the other body. In this sense, although, in general, $\Gamma_p^\alpha \neq \Gamma_p^\beta$, the distinction between these domains becomes immaterial under the preceding assumptions.

Preliminary to discussing integral forms of (7), introduce the notation

$$
A^\alpha(u^\alpha, v^\alpha) = \int_{\Omega^\alpha} \sigma(u^\alpha) \cdot \text{grad} v^\alpha \, dV,
$$

$$
B^\alpha(q^\alpha, v^\alpha) = -\int_{\Gamma_p^\beta} q^\alpha v^\alpha \cdot N^\alpha \, dA,
$$

$$
f^\alpha(v^\alpha) = \int_{\Omega^\alpha} \rho^\alpha b^\alpha \cdot v^\alpha \, dV + \int_{\Gamma_p^\beta} \tilde{t}^\alpha \cdot v^\alpha \, dA.
$$

The spaces of admissible displacements and gaps are respectively defined as

$$
\mathcal{U}^\alpha = \left\{ u^\alpha \in H^1(\Omega^\alpha) \mid u^\alpha = 0 \text{ on } \Gamma_u^\alpha \right\}
$$

and

$$
\mathcal{C}^\alpha_+ = \left\{ v^\alpha \in \mathcal{W}^\alpha \mid v^\alpha \geq 0 \right\},
$$

where $\mathcal{W}^\alpha = H^{1/2}(\Gamma_p^\alpha)$. Likewise, the spaces of admissible pressures and pressure differences are defined as

$$
\mathcal{P}^\alpha = \left\{ p^\alpha \in (\mathcal{W}^\alpha)^* \mid \int_{\Gamma_p^\beta} p^\alpha v^\alpha \, dA \geq 0, \quad \forall v^\alpha \in \mathcal{C}^\alpha_+ ; \right. $$

$$
\left. p^\alpha(X^\alpha) = p^\beta(\pi^\alpha(X^\alpha)), \quad \forall X^\alpha \in \Gamma_p^\alpha \right\}
$$

and

$$
\mathcal{Q}^\alpha = \left\{ q^\alpha \in (\mathcal{W}^\alpha)^* \mid q^\alpha(X^\alpha) = q^\beta(\pi^\alpha(X^\alpha)), \quad \forall X^\alpha \in \Gamma_p^\alpha \right\},
$$

respectively, where $(\mathcal{W}^\alpha)^* = H^{-1/2}(\Gamma_p^\alpha)$. Appendix A contains background information on fractional Sobolev spaces, as well as certain specialized results regarding functions in such spaces.

The Galerkin formulation of the boundary-value problem in (7) requires that the solution $(u^\alpha, p^\alpha) \in \mathcal{U}^\alpha \times \mathcal{P}^\alpha$ satisfy the equations

$$
A^\alpha(u^\alpha, v^\alpha) - f^\alpha(v^\alpha) - B^\alpha(p^\alpha, v^\alpha) = 0,
$$

$$
\sum_{\alpha=1}^{2} B^\alpha(q^\alpha - p^\alpha, X^\alpha + u^\alpha) \geq 0,
$$

for all $v^\alpha \in \mathcal{U}^\alpha$ and $q^\alpha \in \mathcal{P}^\alpha$. In deducing the above equations, explicit use is made of (8-10) and the restrictions on $p^\alpha$ and $q^\alpha$ in (16).
A more concise representation of equations (18) is now proposed. To this end, define the Sobolev product space of admissible displacements $\mathcal{U} = \mathcal{U}^1 \times \mathcal{U}^2$ on $\Omega = \Omega^1 \cup \Omega^2$, equipped with the norm

$$
\|u\|_{(1,\Omega)} = \left[ \sum_{\alpha=1}^{2} \|u^\alpha\|_{(1,\Omega^\alpha)}^2 \right]^{1/2},
$$

where $u = (u^1, u^2) \in \mathcal{U}$. Also, define the product space of normal traces $\mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2$ on $\Gamma_p = \Gamma^1_p \cup \Gamma^2_p$ with the associated norm

$$
\|v\|_{(1/2,\Gamma_p)} = \left[ \sum_{\alpha=1}^{2} \|v^\alpha\|_{(1/2,\Gamma_p^\alpha)}^2 \right]^{1/2},
$$

where $v = (v^1, v^2) \in \mathcal{W}$, and, likewise, its dual $\mathcal{W}^* = (\mathcal{W}^1)^* \times (\mathcal{W}^2)^*$ associated with the natural norm

$$
\|q\|_{(-1/2,\Gamma_p)} = \sup_{v^\alpha \in \mathcal{W}^\alpha} \frac{\sum_{\alpha=1}^{2} \int_{\Gamma_p^\alpha} q^\alpha v^\alpha dA}{\|v\|_{(1/2,\Gamma_p)}},
$$

where $q = (q^1, q^2) \in \mathcal{W}^*$. Equations (18) can now be rewritten as

$$
A(u, v) - f(v) - B(p, v) = 0, \quad B(q - p, X + u) \geq 0,
$$

where

$$
A(u, v) = \sum_{\alpha=1}^{2} A^\alpha(u^\alpha, v^\alpha), \quad B(q, v) = \sum_{\alpha=1}^{2} B^\alpha(q^\alpha, v^\alpha), \quad f(v) = \sum_{\alpha=1}^{2} f^\alpha(v^\alpha),
$$

and should hold true for all $v \in \mathcal{U}$ and $q \in \mathcal{P}$.

Existence and uniqueness of the solution to equations (22) can be proved by extending the procedure employed in [13] for the Signorini problem. To this end, introduce the Lagrangian functional

$$
L(v, q) = J(v) - B(q, v),
$$

in terms of the potential functional

$$
J(v) = \frac{1}{2} A(v, v) - f(v),
$$

as well as the constrained set of admissible displacements

$$
\mathcal{K} = \{ v \in \mathcal{U} \mid \gamma^\alpha(X^\alpha; v^\alpha, v'^\beta) \in C^1_+ \}.
$$

Several important properties pertaining to the functionals $J(v)$ and $L(v, q)$, and to the constraint space $\mathcal{K}$ are outlined below:
(a) The functional $J(v)$ is convex and lower semi-continuous in $U$. These properties are satisfied within the framework of linear elasticity, subject to realistic choices of the material constants. In fact, here $J(v)$ is strictly convex in $U$, since it has been earlier assumed that all rigid modes are suppressed by Dirichlet boundary conditions. Furthermore, $\inf_{v \in K} J(v) < \infty$, which signifies that the contact state is attainable.

(b) The constraint space $K$ is a closed and convex subset of $U$. With reference to (26), convexity follows from the linearity of $g^a$ in $v$, while closure follows from the non-strict inequality constraint involving $g^a$. Invoking the definition of $P$ in (15,16), equations (12), definitions (13) and (23), and standard properties of closed, convex and pointed cones [14, equation (5.9)], one can equivalently define the constraint space $K$ as

$$K = \left\{ v \in U \mid B(q, X + v) \geq 0, \quad \forall q \in P \right\}. \quad (27)$$

(c) There exist $(u^1, u^2) \in U$, such that $g^a(X^a; u^a, u^b) \geq 0$, i.e., one may find displacements that satisfy the constraint of impenetrability.

(d) There exist $(u^1, u^2) \in U$, such that $g^a(X^a; u^a, u^b) < 0$, i.e., one may find displacements that violate the constraint of impenetrability.

(e) The linear form $B(q, X + v)$ is convex and lower semi-continuous in $U$ for fixed $q \in P$.

(f) The linear form $B(q, X + v)$ is convex and lower semi-continuous in $P$ for fixed $v \in U$.

(g) The functional $L(v, q)$ is Gâteaux differentiable in $v$ for any $q \in P$ and Gâteaux differentiable in $q$ for any $v \in U$.

Any one of the following two additional conditions is also crucial to the analysis of the two-body problem:

(h) For every $v \in H^{1/2}(\Gamma_p)$, there is a $v \in U$ such that the normal trace $\gamma$ of $v$ satisfies $\gamma(v) = v$ on $\Gamma_p$ and there exist constants $c, d > 0$, such that $c \|v\|_{(1,\Omega)} \leq \|v\|_{(1/2,\Gamma_p)} \leq d \|v\|_{(1,\Omega)}$.

(h)′ The normal trace of $v \in U$ spans $H^{1/2}_{00}(\Gamma_p)$, so that $W = H^{1/2}_{00}(\Gamma_p)$. In this case, there exist constants $c, d > 0$, such that $c \|v\|_{(1,\Omega)} \leq \|v\|_{(1/2,\Gamma_p)} \leq d \|v\|_{(1,\Omega)}$, where $\|v\|_{(1/2,\Gamma_p)}$ is the $H^{1/2}_{00}(\Gamma_p)$-norm of $v$. 

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8
The satisfaction of conditions (c)-(g) can be readily verified. Appendix A discusses in detail a problem in which condition (h) is satisfied and which is used in the subsequent analysis of finite element formulations. The appendix also discusses the alternative condition (h)'.

The following theorem provides the basis for the analysis of dual formulations of two-body contact.

**Theorem 2.1** If conditions (a)-(g) and (h) or (h)' are satisfied, the weighted residual problem (22) possesses a unique solution \((u, p) \in \mathcal{U} \times \mathcal{P}\).

**Proof:** Appealing to Theorem 5.1 in [14, p. 68], \(u \in \mathcal{U}\) is a solution of the primal problem

\[
J(u) = \inf_{v \in \mathcal{K}} J(v),
\]

if, and only if, properties (a)–(e) above are satisfied and there exists a \(p \in \mathcal{P}\) such that \((u, p)\) is a saddle point of the Lagrangian functional \(L(v, q)\) defined in (24). However, by Proposition 1.2 in [14, p. 35], the primal problem (28) possesses a unique solution \(u\), since \(\mathcal{K}\) is closed and convex and \(J(v)\) is strictly convex. Hence, there exists a (perhaps non-unique) \(p\), such that \((u, p)\) is a saddle point of \(L(v, q)\). By Proposition 1.6 in [14, p. 169], the solution to the saddle-point problem for \(L(v, q)\) is characterized by the inequalities

\[
A(u, v - u) - f(v - u) - B(p, v - u) \geq 0, \quad \forall v \in \mathcal{U},
\]

\[
B(q - p, X + u) \geq 0, \quad \forall q \in \mathcal{P},
\]

subject to conditions (a) and (e)–(g). Since \(v\) is arbitrary, one may substitute \(v\) for \(v - u\) in (29)\(_1\). Upon further noting that \(-v \in \mathcal{U}\), it is readily concluded that (29)\(_1\) holds true as an equality, hence equations (22) and (29) are identical.

To complete the proof, assume that there does not exist a unique \(p\), such that \((u, p)\) is a solution to (22). This implies that, given such a non-unique \(p\), one may find a \(p' \in \mathcal{P}\), \(p' \neq p\), such that

\[
\sup_{v \in \mathcal{U}} \frac{B(p - p', v)}{\|v\|_{(1, \Omega)}} = 0.
\]

In this case, \(q = p - p'\) can be taken to be an arbitrary member of \(Q = Q^1 \times Q^2\). Noting that range \((v^\alpha \cdot N^\alpha) = \mathcal{W}^\alpha\), and recalling (21) and condition (h) or (h)'\(_1\), it is seen that

\[
\|q\|_{(-1/2, \Gamma_p)} = \sup_{v^\alpha \in \mathcal{W}^\alpha} \frac{\sum_{\alpha=1}^{2} \int_{\Gamma_p} q^\alpha v^\alpha dA}{\|v\|_{(1/2, \Gamma_p)}} \leq \frac{1}{c} \sup_{v \in \mathcal{U}} \frac{B(q, v)}{\|v\|_{(1, \Omega)}}.
\]

It follows from (31) that

\[
\sup_{v \in \mathcal{U}} \frac{B(q, v)}{\|v\|_{(1, \Omega)}} \geq c\|q\|_{(-1/2, \Gamma_p)},
\]
for all \( q \in Q \), which contradicts (30), thus proving the uniqueness of the pressure \( p \). \( \square \)

3 A Discrete Babuška-Brezzi Condition for Two-body Contact

In dual finite element methods for the two-body contact problem, finite-dimensional subspaces \( \mathcal{U}_h \) and \( \mathcal{W}_h \) are constructed in order to approximate \( \mathcal{U} \) and \( \mathcal{W} \), respectively. Subsequently, an admissible pressure space \( \mathcal{P}_h \subset \mathcal{P} \) is constructed from \( \mathcal{W}_h^* \) as in (16). Also, the discrete counterpart of the constraint space \( \mathcal{K} \) is defined as

\[
\mathcal{K}_h = \left\{ v_h \in \mathcal{U}_h \mid B(q_h, X + v_h) \geq 0, \ \forall \ q_h \in \mathcal{P}_h \right\} .
\]  

(33)

The set \( \mathcal{K}_h \) is obviously closed and convex in \( \mathcal{U}_h \). However, since \( \mathcal{P}_h \) is, in general, chosen independently of \( \mathcal{U}_h \), the equivalence between the definitions of \( \mathcal{K}_h \) in (26) and (27) breaks down, therefore Theorem 2.1 is not directly applicable to the finite-dimensional problem. Instead, the following theorem is established:

**Theorem 3.1** Let conditions (a)–(g) of Theorem 2.1 apply to the Lagrangian functional \( L(v, q) \) of (24), where \( (v, q) \in \mathcal{U} \times \mathcal{P} \), and assume that \( \mathcal{U}_h, \mathcal{P}_h \) are finite-dimensional subspaces of \( \mathcal{U}, \mathcal{P} \), respectively. Then, the saddle-point problem for \( L(v_h, q_h) \), where \( (v_h, q_h) \in \mathcal{U}_h \times \mathcal{P}_h \) possesses a unique solution \( (u_h, p_h) \) if, and only if, there exists a constant \( \alpha_h > 0 \), such that

\[
\sup_{v_h \in \mathcal{U}_h} \frac{B(q_h, v_h)}{\|v_h\|_{(1, \Omega)}} \geq \alpha_h \|q_h\|_{(-1/2, \Gamma_p)},
\]

(34)

for all \( q_h \in Q_h \).

**Proof:** Since \( \mathcal{K}_h \) is a closed convex subset of \( \mathcal{K} \), the primal problem

\[
J(u_h) = \inf_{v_h \in \mathcal{K}_h} J(v_h)
\]

(35)

possesses a unique solution \( u_h \). Also, given that \( \mathcal{U}_h \subset \mathcal{U} \) and \( \mathcal{P}_h \subset \mathcal{P} \), one may appeal to Proposition 1.6 in [14, p. 169], to conclude that the solution \( (u_h, p_h) \) to the saddle-point problem for \( L(v_h, q_h) \), if one exists, is described by

\[
A(u_h, v_h - u_h) - f(v_h - u_h) - B(p_h, v_h - u_h) \geq 0, \ \forall \ v_h \in \mathcal{U}_h , \]

\[
B(q_h - p_h, X + u_h) \geq 0, \ \forall \ q_h \in \mathcal{P}_h .
\]

(36)
Assume now that equation (34) is not satisfied. Then, since both $\mathcal{P}_h$ and $\mathcal{U}_h$ are finite-dimensional, there exists a non-zero $\tilde{q}_h \in \mathcal{Q}_h$, such that

$$B(\tilde{q}_h, \upsilon_h) = 0 \quad (37)$$

Choose $f(\upsilon_h)$ such that $p_h > 0$ everywhere on $\Gamma_p$. Then, $(\upsilon_h, p_h + \omega \tilde{q}_h)$ is also a solution, where $\omega$ is a scaling parameter chosen to ensure that $p_h + \omega \tilde{q}_h > 0$ everywhere on $\Gamma_p$. Hence, the solution to (36) is not unique. If $p_h$ vanishes on parts of $\Gamma_p$, then the previous argument is applied to the part of $\Gamma_p$ where $p_h > 0$.

Conversely, assume that (34) is satisfied and let $\upsilon_h$ be the solution to (35). Then, by the Kuhn-Tucker Theorem and its corollaries [15, Section 28], there exist coefficients $p_h$ such that (36) is satisfied. Moreover, these coefficients are unique, since $J(\upsilon_h)$ is strictly convex (hence $\upsilon_h$ in (35) is uniquely determined) and equation (36), is linear in $p_h$.

Two general results are now introduced in order to assess the satisfaction or failure of condition (34) for the planar two-body contact problem:

**Proposition 3.1** Consider a finite element discretization $\mathcal{S}_h = \{(\upsilon_h, q_h) \in \mathcal{U}_h \times \mathcal{Q}_h\}$ and assume that smooth contact is enforced over a region $\Gamma_p$ of length $L$. Let $\upsilon_h^\alpha(s) = \sum_{i=1}^{N^\alpha} \upsilon_i^\alpha \Phi_i^\alpha(s)$, where $\Phi_i^\alpha(s)$, $I = 1, N^\alpha$, are piecewise polynomials in the arc length $s$ along $\Gamma_p^\alpha$, and $\upsilon_i^\alpha$ are arbitrary constant vectors. Also, suppose there exists a $\tilde{q}_h(s) \in \mathcal{Q}_h$, such that $B^\alpha(\tilde{q}_h^\alpha(s), \upsilon_i^\alpha \Phi_i^\alpha(s)) = 0$ for all $I$, except, possibly, for $I = 1, M^\alpha$ and $I = (N^\alpha - M^\alpha), N^\alpha$, where $M^\alpha < N^\alpha$ is a positive integer independent of $h$. Furthermore, suppose that $|\tilde{q}_h^\alpha(0)|$, $|\tilde{q}_h^\alpha(L)|$, and $\|\tilde{q}_h^\alpha\|_{(0, \Gamma_p)}$ are all of order $o(1)$ with respect to $h$. Then, the discretization $\mathcal{S}_h$ fails condition (34) of Theorem 3.1.

**Proof:** Consider first the case $M^\alpha = 1$. It follows from the stated assumptions of the proposition and the Cauchy-Schwartz inequality that

$$B^\alpha(\tilde{q}_h^\alpha, \upsilon_h^\alpha) = \int_{\sigma^\alpha_{\tilde{q}_h}} \tilde{q}_h^\alpha(s) \upsilon_h^\alpha(s) \, ds + \int_{\sigma^\alpha_{\upsilon_h}} \tilde{q}_h^\alpha(s) \upsilon_h^\alpha(s) \, ds$$

$$= \upsilon_1^\alpha \int_{\sigma^\alpha_{\tilde{q}_h}} \tilde{q}_h^\alpha(s) \Phi_1^\alpha(s) \, ds + \upsilon_2^\alpha \int_{\sigma^\alpha_{\upsilon_h}} \tilde{q}_h^\alpha(s) \Phi_2^\alpha(s) \, ds \quad (38)$$

$$\leq \upsilon_1^\alpha o(h) + \upsilon_2^\alpha o(h),$$

where $\upsilon_h^\alpha = -\upsilon_h^\alpha \cdot N^\alpha$ and $\sigma_i^\alpha = \text{supp} (\Phi_i^\alpha) \cap \Gamma_p^\alpha$. Given that the fraction on the left-hand side of (34) is independent of the order of $\upsilon_h$, one may, without any loss of generality, set $|\upsilon_i^\alpha| = |\upsilon_{N^\alpha}^\alpha| = 1$, so that $B^\alpha(\tilde{q}_h^\alpha, \upsilon_h^\alpha) \leq o(h)$. Since, in addition, Proposition B.1 in
Appendix B implies that $\|v_h\|_{(1/2, \Gamma_p)}$ is of order $o(1)$ and, by assumption, $\|\tilde{q}_h\|_{(0, \Gamma_p)}$ is also of order $o(1)$, it follows for this choice of $\tilde{q}_h$ that

$$\sup_{v_h \in \mathcal{W}_h} \frac{B(q_h, v_h)}{\|q_h\|_{(0, \Gamma_p)} \|v_h\|_{(1/2, \Gamma_p)}} \leq o(h) .$$

(39)

Invoking condition (h) or (h)' in Theorem 2.1, it follows that

$$\sup_{v_h \in \mathcal{W}_h} \frac{B(\tilde{q}_h, v_h)}{\|q_h\|_{(0, \Gamma_p)} \|v_h\|_{(1, \Omega)}} \leq o(h) ,$$

(40)

therefore the discretization $\mathcal{S}_h$ fails condition (34).

Consider now the case $M^\alpha > 1$. In analogy with the case $M^\alpha = 1$, here

$$B^\alpha(\tilde{q}_h^\alpha, v_h^\alpha) = \int_{\cup_{I=1}^{M^\alpha}} \tilde{q}_h^\alpha(s) v_h^\alpha(s) ds + \int_{\cup_{I=1}^{N^\alpha}} \tilde{q}_h^\alpha(s) v_h^\alpha(s) ds$$

$$= \sum_{I=1}^{M^\alpha} \sum_{I=1}^{N^\alpha} v_I^\alpha \tilde{q}_h^\alpha(s) \Phi_I^\alpha(s) ds + \sum_{I=1}^{N^\alpha} v_I^\alpha \tilde{q}_h^\alpha(s) \Phi_I^\alpha(s) ds$$

(41)

$$\leq \sum_{I=1}^{M^\alpha} v_I^\alpha o(h) + \sum_{I=1}^{N^\alpha} v_I^\alpha o(h) .$$

Since $M^\alpha$ is independent of $h$, the condition $B^\alpha(\tilde{q}_h^\alpha, v_h^\alpha) \leq o(h)$ holds true when taking $v_I^\alpha$ to be of order $o(1)$ for $I = 1, M^\alpha$ or $I = (N^\alpha - M^\alpha), N^\alpha$. Again, no loss of generality is incurred by this assumption, as argued in the case $M^\alpha = 1$. This, in turn, implies that a suitable combination of any such $v_I^\alpha$ satisfies the requirements of Proposition B.1, hence $\|v_h\|_{(1/2, \Gamma_p)}$ is always of order $o(1)$. Therefore, for this choice of $\tilde{q}_h$ one again finds that

$$\sup_{v_h \in \mathcal{W}_h} \frac{B(q_h, v_h)}{\|q_h\|_{(0, \Gamma_p)} \|v_h\|_{(1, \Omega)}} \leq o(h) ,$$

(42)

which completes the proof. ■

**Proposition 3.2** Consider a finite element discretization $\mathcal{S}_h = \{(v_h, q_h) \in \mathcal{U}_h \times \mathcal{Q}_h\}$. Suppose that for any given function $q_h \in \mathcal{Q}_h$, one can construct a function $\tilde{v}_h \in \mathcal{U}_h$, such that

$$\frac{B(q_h, \tilde{v}_h)}{\|\tilde{v}_h\|_{(0, \Gamma_p)}} \geq C \|q_h\|_{(0, \Gamma_p)} ,$$

(43)

where $\tilde{v}_h = \gamma(\tilde{v}_h)$ and $C(> 0)$ is independent of $h$. Also, let the inequality

$$\|v_h\|_{(1/2, \Gamma_p)} \leq \kappa h^{-1/2} \|v_h\|_{(0, \Gamma_p)} ,$$

(44)

hold true for all $v_h \in \mathcal{W}_h$. Then, the discretization $\mathcal{S}_h$ satisfies (34) with $\alpha_h$ of order at least $o(h^{1/2})$. 

12
Proof: Given that for any \( q_h \) there exists a \( \bar{v}_h \) such that (43) holds, inequality (44) implies that

\[
\frac{B(q_h, \bar{v}_h)}{\|q_h\|_{(0, \Gamma_p)} \|\bar{v}_h\|_{(1/2, \Gamma_p)}} \geq \frac{C}{\kappa} h^{1/2}.
\]  
(45)

Invoking condition (h) or (h)' of Theorem 2.1, it follows that

\[
\sup_{\nu_h \in U_h} \frac{B(q_h, \nu_h)}{\|q_h\|_{(0, \Gamma_p)} \|\nu_h\|_{(1, \Omega)}} \geq \frac{C}{\kappa} h^{1/2},
\]

which completes the proof. Conditions under which inequality (44) holds true are discussed in Appendix C. ■

With Propositions 3.1 and 3.2 at one’s disposal, the analysis of particular discretizations is reduced to either determining a (near) singularity-inducing \( q_h \in \mathcal{Q}_h \) or verifying that there exists a singularity-preventing \( \bar{v}_h \in U_h \) for any particular \( q_h \in \mathcal{Q}_h \). Finite element formulations that satisfy the conditions of Proposition 3.2 lead to stable approximations at least when \( h \) is bounded away from zero.

It has been already established in a broad setting of dual finite element methods for constrained problems that condition (34) is intimately connected with the balance between constraints and associated degrees-of-freedom, see [16, 17]. In particular, let \( r_h \) be the ratio of degrees-of-freedom to constraints, namely

\[
r_h = \frac{\dim W_h}{\dim \mathcal{Q}_h}.
\]

(47)

For a discrete solution to be algebraically stable in the sense of boundedness for the spectral radius of the associated Hessian, the condition \( r_h \geq 1 \) should be satisfied. Furthermore, an optimal discrete approximation is heuristically defined as one in which \( r_h \) is equal to the corresponding ratio \( r \) of the continuum problem, see [1, Section 4.3.7]. For the Signorini problem, the optimal constraint ratio is \( r = 1 \), because, upon taking the index \( \alpha = 1 \) to correspond to the deformable body, every point of \( \Gamma^\alpha_p \) possesses one constraint \( g^\alpha = 0 \) and one degree-of-freedom \( v^\alpha \). By the same token, the optimal constraint ratio for the two-body problem is \( r = 2 \). Indeed, taking into account (11), every point of \( \Gamma_p \) still has one independent constraint \( g^\alpha = g^\beta = 0 \) and two degrees-of-freedom \( v^\alpha \) and \( v^\beta \).

4 Convergence and Error Analysis

The present section includes a convergence analysis, subject to the conditions of Theorem 3.1, as well as to the regularity of the exact solution and the properties of the finite element approximation.
Theorem 4.1 Let the conditions of Theorem 3.1 be satisfied. Then, there exist positive constants $C_1$, $C_2$, $C_3$ and $C_4$, such that the absolute error of the approximate solution $(u_h, p_h) \in U_h \times P_h$ when compared to the exact solution $(u, p) \in U \times P$ may be expressed as

$$
\| u - u_h \|_{(1, \Omega)} \leq C_1 \left[ \| v_h - u \|_{(1, \Omega)} + \| q_h - p \|_{(-1/2, \Gamma_p)} + \frac{1}{\alpha_h} \| u - v_h \|_{(0, \Gamma_p)} \right] \\
+ C_2 \left[ \| p - q_h \|_{(0, \Gamma_p)} \| u - v_h \|_{(0, \Gamma_p)} + B(q - p_h + q_h - p, X + u) \right]^{1/2},
$$

(48)

$$
\| p - p_h \|_{(0, \Gamma_p)} \leq C_3 \| q_h - p \|_{(0, \Gamma_p)} + \frac{C_4}{\alpha_h} \left[ \| u - u_h \|_{(1, \Omega)} + \| q_h - p \|_{(1/2, \Gamma_p)} \right],
$$

for all $v_h \in U_h$, $q_h \in P_h$ and $q \in P$, where $u = (u^1 \cdot N^1, u^2 \cdot N^2)$ and $v_h = (v^1_h \cdot N^1, v^2_h \cdot N^2)$.

**Proof:** The proof is entirely analogous to that of [13, Theorem 4.7] for the Signorini problem, the only difference being in the inclusion of contributions from both bodies in definition of $B(v, q)$ and the admissible spaces. □

The exact solution $(u, p) \in U \times P$ is taken to be sufficiently regular to render the convergence analysis practicable. In particular, it is assumed that

$$
u^\alpha \in H^\tau(\Omega^\alpha) \quad , \quad u^\alpha \cdot N^\alpha \in H^{\tau - 1/2}(\Gamma_p^\alpha) \quad , \quad p^\alpha \in H^{\tau - 3/2}(\Gamma_p^\alpha),
$$

(49)

where $\tau(\geq 2)$ is an integer. In addition, the finite-dimensional approximations $(v_h, q_h) \in U_h \times Q_h$ are taken to be polynomially complete to degrees $k(\geq 1)$ and $l(\geq 0)$, respectively. Furthermore, the finite element meshes are assumed to satisfy two standard assumptions on topological regularity. The first requires that there exist a constant $\ell > 0$, such that for any element $e$,

$$
\frac{h_e}{r_e} \leq \ell,
$$

(50)

where $h_e$ is the “diameter” of element $e$ and $r_e$ the supremum over the radii of all balls inside element $e$, see [18, p. 132]. The second assumption admits the existence of a constant $\nu > 0$, such that for any element $e$,

$$
\frac{h}{h_e} \leq \nu,
$$

(51)

where $h = \sup_e h_e$ [18, p. 140]. The conditions (50) and (51) are not considered practically restrictive. A sequence of spatially refined meshes that satisfy (50) and (51) is said to be quasi-uniform [19, Definition 4.4.13].

The interpolation functions $v_h \in U_h$ are taken to satisfy the following “inverse assumption”: given any $\epsilon > 0$, where $\epsilon < 1$, and any $s > 0$, such that $1 - \epsilon \leq s \leq 1$, there exists a
\[ K > 0 \text{ such that} \]
\[ \| v_h \|_{(1, \Omega)} \leq K h^{s-1} \| v_h \|_{(s, \Omega)} , \tag{52} \]
for all \( v_h \in \mathcal{U}_h \) (see [18, pages 140-142] and [20, page 88]). Theorem 4.5.11 and Remark 4.5.20 in [19] establish that (52) is satisfied for quasi-uniform families of subdivisions of polyhedral domains in \( \mathcal{E}^d \), provided the subject finite element families are conforming with respect to both \( H^1(\Omega) \) and \( H^s(\Omega) \).

Quasi-uniform families of meshes are associated with well-known interpolation properties [13, Theorem 4.3], originally established by Falk [21]. Specifically, given any functions \( v^a \in \mathcal{U}^a \) and \( q^a \in Q^a \) that satisfy the smoothness requirements in (49), corresponding discrete approximations \( v_h \) and \( q_h \) can be found such that
\begin{align}
\| v_h - v \|_{(1, \Omega)} & \leq \kappa_1 h^\mu \| v \|_{(\tau, \Omega)} , \\
\| v_h - v \|_{(0, \Gamma_p)} & \leq \kappa_2 h^{\mu+1/2} \| v \|_{(\tau, \Omega)} , \\
\| q_h - q \|_{(0, \Gamma_p)} & \leq \kappa_3 h^\lambda \| q \|_{(\tau-3/2, \Gamma_p)} , \\
\| q_h - q \|_{(-1/2, \Gamma_p)} & \leq \kappa_4 h^{\lambda+1/2} \| q \|_{(\tau-3/2, \Gamma_p)} ,
\end{align}
where \( \kappa_1, \kappa_2, \kappa_3 \) and \( \kappa_4 \) are positive constants and
\[ \mu = \min(\tau - 1, k) , \quad \lambda = \min(\tau - 3/2, l + 1) . \tag{54} \]

Note that it is possible to find a function \( v_h \) such that conditions (53)\(_{1,2}\) are satisfied simultaneously. This is because the proof of each of these inequalities relies on choosing \( v_h = v \) at each nodal point, which can be accomplished simultaneously in the domain \( \Omega \) and on the boundary \( \partial \Omega \) when considering standard finite element approximation functions.

Taking into account (53), the error estimates (48) of Theorem 4.1 lead to
\begin{align}
\| u - u_h \|_{(1, \Omega)} & \leq o(h^\min(\mu, \lambda+1/2, \mu+1/2, \lambda+\kappa_4/2)) , \\
\| p - p_h \|_{(0, \Gamma_p)} & \leq o(h^\min(\lambda, o(\| u - u_h \|_{(1, \Omega)} - o(\alpha_h), \lambda+1/2 - o(\alpha_h)) . \tag{55} \]

As seen from (55), owing to the interdependence of the errors in \( u_h \) and \( p_h \), given any \( k \) there exists a value of \( l \) beyond which no further improvement in the order of accuracy is achieved for the approximation of either \( u \) or \( p \). Hence, it is possible to identify the minimum degree of polynomial completeness in \( l \) needed for a given \( k \), such that an optimal convergence rate be attained.
5 A Review of Formulations for the Signorini Problem

The pioneering work of Oden, Kikuchi and coworkers [3, 4, 5, 6] focused on the stability and convergence properties of certain contact finite elements for the Signorini problem in the context of reduced-integration penalty methods. As a prelude to the discussion of two-body finite element formulations, the Signorini problem is revisited here with emphasis on unregularized Lagrange multiplier methods involving linear and quadratic domain interpolations. The forthcoming analysis is substantially different in context and methodology from that in [3, 4, 5, 6].

Preliminary to the evaluation of specific interpolations, a canonical Signorini problem is specified. This problem involves frictionless contact between a deformable body, which occupies a square domain $\Omega = \Omega_\square \in \mathcal{E}^2$ of side $L$, with a flat, semi-infinite, stationary and rigid foundation. In particular, contact is assumed to occur simultaneously along an entire edge of $\Omega_\square$, hence here $\Gamma_p = \Gamma_c$, as seen in Figure 1. In addition, the domain $\Omega_\square$ is meshed uniformly using triangular or rectangular elements. Assuming there are $N - 1$ element edges along $\Gamma_p$, a typical such edge is denoted by $\Gamma^J_p$, so that $\Gamma_p = \bigcup_{j=1}^{N-1} \Gamma^J_p$ and $h = \frac{N-1}{L}$.

5.1 Linear domain elements

Let $\Omega_\square$ be meshed uniformly using linear triangular or bilinear square elements, so that in both cases $k = 1$. With reference to Figure 2, proceed to examine the following possible choices for $Q_h$:

(S10D1) Piecewise constant functions over each element edge $\Gamma^J_p$.

(S11D1) Discontinuous piecewise linear functions over each element edge $\Gamma^J_p$.

(S10D2) Piecewise constant functions over each successive pair of element edges $\Gamma^J_p$ and $\Gamma^{J+1}_p$.

(S11C1) Continuous piecewise linear functions over each element edge $\Gamma^J_p$.

The next two propositions establish the stability properties of the resulting contact elements:

**Proposition 5.1** The formulations S10D1 and S11D1 fail condition (34).

**Proof:** Choose $\bar{q}_h = q_J$ on $\Gamma^J_p$, such that

$$q_J = Q(-1)^J,$$

(56)
where $Q(>0)$ is a constant. This implies that

$$
\| \bar{q}_h \|_{(0, \Gamma_p)} = \left[ \sum_{j=1}^{N-1} \int_0^h Q^2 ds \right]^{1/2} = Q L^{1/2}.
$$

(57)

Also, recall that the displacement $\mathbf{v}_h$ along $\Gamma_p$ can be expressed as $\mathbf{v}_h = \sum_{I=1}^N \mathbf{v}_I \Phi_I$ in terms of the linear interpolation functions $\Phi_I$ and the associated degrees of freedom $\mathbf{v}_I$ along the boundary. It is now easily seen that for any contact boundary node $I$ away from the edge of contact (i.e., other than $I = 1$ or $I = N$),

$$
B(\bar{q}_h, \mathbf{v}_I \Phi_I) = - \int_0^h Q(-1)^{I-1} \mathbf{v}_I \cdot \mathbf{N} \frac{s}{h} ds - \int_0^h Q(-1)^{I} \mathbf{v}_I \cdot \mathbf{N} (1 - \frac{s}{h}) ds = 0.
$$

(58)

By virtue of Proposition 3.1, equation (58) implies that the formulation S10D1 fails condition (34).

The preceding selection of piecewise constant $\bar{q}_h$ is also admissible for the formulation S11D1. Hence, this formulation fails condition (34) as well. ■

It is remarked that the instability of formulation S11D1 can be immediately ascertained by means of simple ”constraint counts”, namely by observing that there are more constraints than active equations on $\Gamma_p$, see [16]. However, the failure of formulation S10D1 cannot be predicted using the same argument.

**Proposition 5.2** The formulations S10D2 and S11C1 satisfy condition (34).

**Proof:** Starting with formulation S10D2, admit consecutive node numbering on $\Gamma_p$, such that, without loss of generality, any element of $Q_h$ satisfies $q_h = Q(I+1)/2$ in $\Gamma^I_p \cup \Gamma^{I+1}_p$, for constant $Q(I+1)/2(>0)$ and $I$ odd. Therefore, choosing $\mathbf{v}_h = \sum_{I=1}^N \mathbf{v}_I \Phi_I$, where

$$
\mathbf{v}_I = \begin{cases} 
0 & \text{for } I \text{ odd} \\
-Qh/2 \mathbf{N} & \text{for } I \text{ even}
\end{cases}.
$$

(59)

it follows trivially that

$$
B(q_h, \mathbf{v}_h) = \frac{1}{2} \| q_h \|_{(0, \Gamma_p)}^2
$$

(60)

and

$$
\| \mathbf{v}_h \|_{(0, \Gamma_p)} = \frac{1}{\sqrt{3}} \| q_h \|_{(0, \Gamma_p)}.
$$

(61)

Hence, appealing to Proposition 3.2, it is concluded that the formulation SD10D2 satisfies condition (34).
Turning to formulation S11C1, choose \( \tilde{\mathbf{v}}_I = q_I \mathbf{N} \) for any admissible \( q_h = \sum_{I=1}^{N} q_I \Phi_I \). This leads to
\[
B(q_h, \tilde{\mathbf{v}}_h) = \| \tilde{\mathbf{v}}_h \|^2_{(0, \Gamma_p)} = \| q_h \|^2_{(0, \Gamma_p)},
\]
therefore, again, Proposition 3.2 guarantees that condition (34) is satisfied. ■

Note that the classical one-pass node-on-surface formulation for the Signorini problem corresponds to S11C1, subject to integration of \( B(q_h, \tilde{\mathbf{v}}_h) \) by the trapezoidal rule. In this case, the trapezoidal rule is sufficient for this reduced integration formulation to pass the constant pressure “contact patch test”, see [22].

5.2 Quadratic domain elements

Let the rectangular domain \( \Omega_\square \) of Figure 1 be discretized this time using 6-node triangular or 8/9-node rectangular elements, such that \( k = 2 \). Figure 3 illustrates the following choices for \( Q_h \):

(S20D1) Piecewise constant functions over each element edge \( \Gamma_p^J \).
(S21D1) Discontinuous piecewise linear functions over each element edge \( \Gamma_p^J \).
(S21C1) Continuous piecewise linear functions over each element edge \( \Gamma_p^J \).
(S22C1) Continuous piecewise quadratic functions over each element edge \( \Gamma_p^J \).

**Proposition 5.3** The formulation S21D1 fails condition (34).

**Proof:** As in Proposition 5.1, choose \( \tilde{q}_h \) to be alternating on successive domains \( \Gamma_p^J \) and invoke Proposition 3.1. ■

**Proposition 5.4** The formulations S20D1, S21C1 and S22C1 satisfy condition (34).

**Proof:** For formulation S20D1, in which \( q_h = \sum_{I=1}^{N} q_I \Phi_I \), choose
\[
\tilde{\mathbf{v}}_{2J} = q_J \mathbf{N}, \quad \tilde{\mathbf{v}}_{2J-1} = \tilde{\mathbf{v}}_{2J+1} = \mathbf{0},
\]
on edge \( \Gamma_p^J \), where, again, consecutive node numbering is assumed on the contact surface. In this case, one can easily establish that
\[
B(q_h, \tilde{\mathbf{v}}_h) = \frac{4}{3} \| q_h \|^2_{(0, \Gamma_p)}
\]
and
\[
\| \tilde{\mathbf{v}}_h \|_{(0, \Omega)} = \sqrt{15} \| q_h \|_{(0, \Gamma_p)}.
\]
Hence, by Proposition 3.2, formulation S20D1 satisfies condition (34).

Letting \( \tilde{\mathbf{v}} = q_h \mathbf{N} \) on \( \Gamma_p \), the proofs for formulations S21C1 and S22C1 are obtained in analogy to that for formulation S10D2 in Proposition 5.2. ■

Although at first sight the result for formulation S20D1 appears to be in conflict with that reported in [3, p. 8, item 13], it should be noted that the latter also assumes reduced integration for the calculation of the bilinear form \( B(q_h, \mathbf{v}_h) \).

## 5.3 Summary

The results of the preceding analysis of finite element formulations of the Signorini problem are summarized in the following table:

<table>
<thead>
<tr>
<th>Method</th>
<th>( q_h )</th>
<th>( \psi_h )</th>
<th>Continuity</th>
<th># Edges</th>
<th>Constraint ratio ( r )</th>
<th>LBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>S10D1</td>
<td>1</td>
<td>n</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>Fail</td>
</tr>
<tr>
<td>S11D1</td>
<td>1</td>
<td>n</td>
<td>1</td>
<td>1</td>
<td>1/2</td>
<td>Fail</td>
</tr>
<tr>
<td>S10D2</td>
<td>1</td>
<td>y</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>Pass</td>
</tr>
<tr>
<td>S11C1</td>
<td>1</td>
<td>y</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Pass</td>
</tr>
<tr>
<td>S20D1</td>
<td>2</td>
<td>n</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>Pass</td>
</tr>
<tr>
<td>S21D1</td>
<td>2</td>
<td>n</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>Fail</td>
</tr>
<tr>
<td>S21C1</td>
<td>2</td>
<td>y</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>Pass</td>
</tr>
<tr>
<td>S22C1</td>
<td>2</td>
<td>y</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>Pass</td>
</tr>
</tbody>
</table>

Formulations S11C1 and S22C1 are optimal in the sense of the constraint ratio defined in equation (47). Formulations S10D2 and S21C1 are optimal in the sense of convergence discussed in Section 4.

## 6 Two-body Contact Formulations

### 6.1 The reference analytical configuration

The two-body contact problem is inherently complicated by the arbitrary alignment of elements on the interface. In order to circumvent this difficulty, an analysis for the two-body problem is conducted here by assuming an \textit{a priori} configuration (termed the “reference analytical configuration”), as illustrated in Figure 4. This configuration is intended to simplify the analysis without compromising the main features of the discrete two-body
problem. The reference analytical configuration consists of two identical square regions $\Omega_1$ and $\Omega_2$ with side length $L$. During contact, one region is offset with respect to the other by $\gamma L$, where $\gamma > 0$ is an irrational number. Each region is meshed in such a way that there are $N$ equally-spaced nodes per side so that $h = \frac{L}{N-1}$. The choice of irrational $\gamma$ guarantees that the nodes on the contact surfaces remain offset from each other by a value $\beta h$, where $\beta > 0$ is again irrational and depends on $h$. This further implies that there is never a case of node-on-node contact under uniform mesh refinement.

An analysis of the two-body problem for the reference analytical configuration can be used to conclusively reject formulations that fail condition (34) of Theorem 3.1. At this stage, it has not been established that satisfaction of (34) for this configuration alone guarantees stability for all configurations. However, the reference analytical configuration and the Signorini problem can be viewed as two “extreme cases” of the two-body problem. In the former, the meshing of the two bodies is assumed uniform with no node-on-node conditions. On the other hand, the Signorini problem may be viewed as representing either the case of uniform node-on-node contact or, in a limiting sense, the case of contact between a very coarse and a very fine mesh. Satisfaction of (34) for both extremes provides a reasonable indication (albeit not a proof!) of stability for the general discrete two-body problem.

6.2 “Node-on-surface” formulations

The development of stable and accurate dual formulations for the two-body contact problem is substantially complicated by the presence of separate displacement and, possibly, pressure interpolations on $\Gamma_1^\alpha$ and $\Gamma_2^\beta$. This difficulty is, to a great extent, circumvented by “node-on-surface” formulations, in which the bilinear form $B(q_h, v_h)$ is a priori replaced by a nodally-based approximation. To elaborate, let $v_h^\alpha(s^\alpha) = \sum_{i=1}^{N^\alpha} v_i^\alpha \Phi_i^\alpha(s^\alpha)$ and $q_h^\beta(s^\beta) = \sum_{j=1}^{M} q_j^\beta \Psi_j(s^\alpha)$, where $N^\alpha$ is the number of nodes on $\Gamma_1^\alpha$ and $M$ the total number of pressure degrees-of-freedom on both $\Gamma_1^\alpha$ and $\Gamma_2^\beta$. With the aid of (13)\textsubscript{2} and (23)\textsubscript{2}, it is seen that

$$B(q_h, v_h) = -2 \sum_{\alpha=1}^{2} \left[ \sum_{J=1}^{M} q_J^{\beta} \sum_{I=1}^{N^\alpha} \left( \int_{\Gamma_1^\beta} \Psi_J^\alpha \Phi_I^\alpha \, ds \right) \cdot v_I^\alpha \right]. \tag{66}$$

Pressure continuity on the contact interface is guaranteed by constructing the pressure interpolations such that

$$\pi^\alpha(X^\alpha(s^\alpha)) = X^\beta(s^\beta) \implies \Psi_J^\beta(s^\beta) = \Psi_J^\alpha(s^\alpha). \tag{67}$$
In general, the bilinear form $B(q_h, v_h)$ of (66) can be approximately computed using a quadrature rule, namely

$$B(q_h, v_h) = - \sum_{a=1}^{2} \sum_{J=1}^{M} q_J \sum_{l=1}^{L_a} \left( w^\alpha(l) \Psi^\alpha_I(s^\alpha_I) \Phi^\alpha_I(s^\alpha_I) \mathbf{N}^\alpha(s^\alpha_I) \right) \cdot \mathbf{v}_I,$$  

(68)

where $s^\alpha_I$ denotes the surface coordinate of quadrature point $l$ on $\Gamma^\alpha_p$ and $w^\alpha(l)$ the corresponding integration weight. In node-on-surface formulations, it is assumed that there exists a special quadrature rule in which $L^1 = L^2 = M$ and $w^\alpha(l) = w^\beta(l) = w(l)$ for $l = 1, 2, \ldots, M$. Furthermore, it is assumed that the pressure interpolation functions satisfy $\Psi^\alpha_I(s^\alpha_I) = \delta_{Jl}$ and the integration points coincide with element nodal points on $\Gamma^\alpha_p$ and $\Gamma^\beta_p$, hence $L^\alpha = N^\alpha + N^\beta$. With the above assumptions, equation (68) takes the form

$$B(q_h, v_h) = - \sum_{J=1}^{M} Q_J \sum_{a=1}^{2} \sum_{l=1}^{N^\alpha} \left( \Phi^\alpha_I(s^\alpha_J) \mathbf{N}^\alpha(s^\alpha_J) \right) \cdot \mathbf{v}_I,$$  

(69)

where $Q_J = w(J)q_J$. Therefore, a “node-on-surface” method approximates the pressure $q_h$ over the contact surface with a discrete set of nodal forces $Q_J$ applied at points with coordinates $s^\alpha_J$ and $s^\beta_J$. Equation (69) defines the “two-pass” version of the node-on-surface formulation in which discrete nodal forces $Q_J$ are generated at quadrature points on both surfaces.\(^1\) In the “one-pass” formulation with $\Gamma^\alpha_p$ being the “slave” surface, $L^\alpha = L^\beta = N^\alpha = M$ and equation (69) still holds true.

The following proposition addresses the stability of node-on-surface formulations:

**Proposition 6.1** Consider a finite element discretization $\mathcal{S}_h = \{ (v_h, q_h) \in \mathcal{U}_h \times \mathcal{Q}_h \}$ and assume that smooth contact is enforced over a region $\Gamma_p$ of length $L$. Also, assume that a node-on-surface formulation is obtained as in (69), such that there exists a set of discrete forces $\bar{Q}_J$ for which

$$\sum_{J=1}^{M} \left[ \bar{Q}_J \sum_{a=1}^{2} \sum_{l=1}^{N^\alpha} \left( \Phi^\alpha_I(s^\alpha_J) \mathbf{N}^\alpha(s^\alpha_J) \right) \cdot \mathbf{v}_I \right] = 0,$$  

(70)

for all $v_h$ except, possibly, for a finite and independent of $h$ number of $v^\alpha_I$ corresponding to nodes at the edges of the contact regions $\Gamma^\alpha_p$. Then, any pressure interpolation that corresponds to the discrete forces $\bar{Q}_J$ fails condition (34) of Theorem 3.1.

\(^1\)In the special case of node-on-node contact, the number of quadrature points (thus, also the number of nodal forces) $M$ must be reduced by the number of aligned pairs of nodes from the two surfaces.
Proof: If pressure interpolation functions $\Psi^\alpha_J$ corresponding to the node-on-surface forces $Q_J$ exist, then, given any $Q_J$, there exist pressure variables $q_J$, such that

$$
\sum_{\alpha=1}^2 \left[ \sum_{J=1}^M \sum_{I=1}^{N^\alpha} \left( \Phi^\alpha_I(s^\alpha_J) N^\alpha_I(s^\alpha_J) \right) \cdot v^\alpha_I \right] = \sum_{\alpha=1}^2 \left[ \sum_{J=1}^M \sum_{I=1}^{N^\alpha} \left( \sum_{l=1}^{L^\alpha} w^\alpha(l) \Psi^\alpha_J(s^\alpha_J) \Phi^\alpha_I(s^\alpha_J) N^\alpha_I(s^\alpha_J) \right) \cdot v^\alpha_I \right],
$$

(71)

for all $v^\alpha_I$. Given that there exist $\tilde{Q}_J$ such that, for every $v^\alpha_I$, except, possibly, a finite number associated with the boundary of contact, condition (70) holds true, then, by (71), there exist pressures $\tilde{q}_J$ such that, except for, possibly, a finite number of boundary nodes, the following equation holds:

$$
\sum_{\alpha=1}^2 \sum_{J=1}^M \sum_{I=1}^{N^\alpha} \left( \sum_{l=1}^{L^\alpha} w^\alpha(l) \Psi^\alpha_J(s^\alpha_J) \Phi^\alpha_I(s^\alpha_J) N^\alpha_I(s^\alpha_J) \right) \cdot v^\alpha_I = 0.
$$

(72)

As nodal variables $v^\alpha_I$ are associated with boundary $\Gamma^1_p$ or $\Gamma^2_p$, it is clear from (66) that

$$
B^\alpha(\tilde{q}, \Phi^\alpha_J v^\alpha_I) = 0,
$$

(73)

except, possibly, for some nodes at the boundary of contact. Then, by Proposition 3.1, the formulation fails the LBB condition. ■

The preceding proposition is utilized for the analysis of specific node-on-surface algorithms in Appendix D.

6.3 One-pass vs. two pass formulations

One- or two-pass formulations can be employed for either node-on-surface or pressure interpolation methods. In a one-pass formulation, the discrete forces or the pressure interpolations are constructed based solely on the mesh of one of the two surfaces (termed the “slave” surface). As a result, such formulations suffer from inherent geometric bias associated with the choice of the slave surface. An immediate consequence of this bias is a potentially unfavorable constraint ratio (47), when contact occurs between a finely meshed slave surface compared to the opposite (“master”) surface. In such a case, the constraint ratio is

$$
\tau = \frac{N_{\text{master}} + N_{\text{slave}}}{N_{\text{slave}}} \leq 1.
$$

(74)

Consequently, the slave surface will be “locked” to the master surface. This is undesirable (albeit not fatal), because in effect it sacrifices the fine interpolation of the slave surface. The reverse scenario yields contact between a coarsely meshed slave and a finely meshed
master surface. Here, the constraint is grossly underenforced, leading to potentially large local penetration of the meshes. Here, the constraint ratio satisfies

\[ r = \frac{N_{\text{master}} + N_{\text{slave}}}{N_{\text{slave}}} \gg 2. \]  

(75)

In problems with gross sliding and where the contacting meshes may be locally refined independently of each other, it is impractical to choose a primary surface that will avoid either of the above two scenarios. This indicates that one-pass methods are less than ideal for general two-body contact.

6.4 Analysis of specific contact schemes

In this section, two-body contact elements are analyzed using the reference analytical configuration and Propositions 3.2 and 3.1. The contact element formulations analyzed here are:

(T11C1) Linear domain elements with one-pass continuous linear pressure over each element edge (Figure 5).

(T1N1) Linear domain elements with one-pass node-on-surface resultants (Figure 6).

(T11C2) Linear domain elements with two-pass continuous linear pressure over each contact segment (Figure 7).

(T1N2) Linear domain elements with two-pass node-on-surface resultants (Figure 8).

(T21C1) Quadratic domain elements with one-pass continuous linear pressure over each element edge (Figure 9).

(T22C1) Quadratic domain elements with one-pass continuous quadratic pressure over each element edge (Figure 10).

(T2N1) Quadratic domain elements with one-pass node-on-surface resultants (Figure 11).

(T2N2) Quadratic domain elements with two-pass node-on-surface resultants (Figure 12).

(T21C2) Quadratic domain elements with two-pass continuous linear pressure over each contact segment (Figure 13).

In the case of frictionless two-body contact, a patch test has been proposed in [22] to assess the ability of a dual formulation to exactly reproduce a constant pressure solution.
As demonstrated in [22], the node-on-surface method T1N2 passes this test, while the methods T1N1, T2N1 and T2N2 do not pass it. In addition, Propositions D.1 and D.3 establish that methods T1N2 and T2N2 fail the LBB condition.

On the other hand, formulations that employ pressure interpolation, such as T11C1, T11C2, T21C1, T22C1, and T21C2, automatically pass the contact patch test conditional upon sufficiently accurate integration of the boundary integrals on the contact surface. This, in general, necessitates the introduction of integration cells (contact segments) defined by means of projections of the boundary meshes from one side to the other. Such integration cells may be computationally expensive and difficult to define robustly, especially in three-dimensions. The one-pass pressure interpolations methods T11C1, T21C1, and T22C1 satisfy the LBB condition, as shown in the following proposition:

**Proposition 6.2** Consider any one-pass pressure interpolation method which, when restricted to the Signorini problem by making the master surface rigid, satisfies the LBB condition. Then, such a formulation also passes the LBB condition for the two-body problem, upon sufficiently accurate integration of contact boundary terms.

**Proof:**
Consider a one-pass formulation using pressure interpolations, such as those in methods T11C1, T21C1, and T22C1. Also, let the Signorini problem resulting from rigidifying the master surface $\beta$ in the above formulations satisfy the LBB condition. Returning to the two-body problem, given any $q_h$, choose $\vec{v}_h = 0$ and $\vec{v}_h^\alpha = \vec{v}_h^\alpha$, where $\vec{v}_h^\alpha$ is the displacement constructed on the slave surface $\alpha$ to satisfy the LBB condition in the Signorini problem. Then one has

$$B^\alpha(q_h, \vec{v}_h^\alpha) = B(q_h, \vec{v}_h)^{\text{Signorini}},$$

and, similarly,

$$\|\vec{v}_h\|_{(1,\Omega)} = \|\vec{v}_h\|_{(1,\Omega^\alpha)} = \|\vec{v}_h\|^\alpha_{\text{Signorini}}$$

and

$$\|q_h\|_{(-1/2,\Gamma_p)} = \|q_h\|^\alpha_{\text{Signorini}}.$$

Therefore, for any $q_h$ one may choose $\vec{v}_h$, such that

$$\frac{B(q_h, \vec{v}_h)}{\|q_h\| \|\vec{v}_h\|} = \frac{B(q_h, \vec{v}_h)}{\|q_h\| \|\vec{v}_h\|^\alpha_{\text{Signorini}}} > \alpha_h,$$

where $\alpha_h > 0$ is obtained from the analysis of the corresponding Signorini problem. ■
In two-pass algorithms, the pressure interpolation is obtained using information from both surfaces. In method T11C2 each node along the contact surface is associated with a nodal pressure value, and the pressure is interpolated linearly between the nodal pressure values. Proposition E.1 in Appendix E demonstrates that this formulation fails the LBB condition for the reference analytical configuration. Likewise, in method T21C2 each node along the contact surface is again associated with a nodal pressure value, and the pressure is again interpolated linearly between the nodal pressure values. However, in contrast with method T11C2, here the domain interpolation is piecewise quadratic. Proposition E.2 in Appendix E demonstrates that method T21C2 also fails the LBB condition for the reference analytical configuration. The failure of methods T1N2, T2N2, T11C2, and T21C2 can be easily explained as a case of overconstraining on the contact interface, sometimes referred to as surface locking [2]. A simple geometric argument reveals that satisfaction of all impenetrability constraints in equality form for methods T11C2 and T1N2 would generally require that the contact interface be globally flat, whereas the same constraints for methods T2N2 and T21C2 would necessitate that the contact interface be globally quadratic. Such locking behavior can be easily detected by noting that the constraint ratio \( r \) for these formulations satisfies \( r = 1 \). Having as many constraints as equations to satisfy them implies that there is a restricted set of admissible configurations for the interface. Such configurations are controlled by the domain interpolation of the two bodies. This, in general, implies surface locking, as the contact boundaries of the two bodies attempt to adjust to these configurations regardless of the kinetics of the underlying boundary-value problem.

As in the case of the Signorini problem, the results of the preceding analysis are summarized in the following table:
An analysis of dual two-body contact formulations

<table>
<thead>
<tr>
<th>Method</th>
<th>$v_h$</th>
<th>$q_h$</th>
<th>Constraint Ratio $r$</th>
<th>Patch Test</th>
<th>LBB</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>Continuity</td>
<td>$l$</td>
<td># Passes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T1C1</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>&gt;1</td>
<td>Pass</td>
</tr>
<tr>
<td>T1N1</td>
<td>1</td>
<td>-</td>
<td>1</td>
<td>&gt;1</td>
<td>Fail</td>
</tr>
<tr>
<td>T1C2</td>
<td>1</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>Pass</td>
</tr>
<tr>
<td>T1N2</td>
<td>1</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>Pass</td>
</tr>
<tr>
<td>T2C1</td>
<td>2</td>
<td>-</td>
<td>1</td>
<td>&gt;1</td>
<td>Pass</td>
</tr>
<tr>
<td>T2N1</td>
<td>2</td>
<td>-</td>
<td>1</td>
<td>&gt;1</td>
<td>Fail</td>
</tr>
<tr>
<td>T2N2</td>
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<td>-</td>
<td>2</td>
<td>1</td>
<td>Fail</td>
</tr>
<tr>
<td>T2C2</td>
<td>2</td>
<td>-</td>
<td>2</td>
<td>1</td>
<td>Pass</td>
</tr>
</tbody>
</table>

7 An Alternative Pressure Interpolation for Two-body Contact

A close inspection of the reasons behind the failure of two-pass algorithms (both pressure- and nodally-based) to satisfy the LBB condition reveals that the controlling factor is the number and distribution of constraints. In nodally-based schemes, the task of reducing the number of constraints while still passing the contact patch test appears rather formidable. On the other hand, in pressure-based interpolations, there exists more flexibility in reducing the number of constraints. An investigation of the Signorini problem provides helpful insight towards a possible solution of the overconstraining in two-pass formulations. Indeed, in the former, assuming piecewise linear domain interpolation for the displacements ($k = 1$) and piecewise constant pressure interpolation over each edge ($l = 0$), the LBB condition fails, as shown in Proposition 5.1. However, when the pressure is assumed piecewise constant over a pair of contiguous element edges, the LBB condition is satisfied. Specifically, as argued in Proposition 5.2, given any $q_h$, the center nodes of each pair of element edges are free to generate a displacement field $\tilde{v}_h$ that bounds the bilinear form $B$ away from zero. An analogous path is followed in constructing alternative two-pass pressure interpolations for the two-body problem.

Figures 14 and 15 depict contact elements T11MC2 and T21MC2, respectively. In these elements, the pressure fields are taken to be continuous and piecewise linear over a pair of contiguous (linear or quadratic) elements edges, such that the center nodes of opposing pairs are not directly associated with pressure variables. In particular, with reference to
Figure 14, let a pair of elements with domains $\Omega_{4I-2}$ and $\Omega_{4I}$ and normal displacement degrees-of-freedom $v_{4I-2}$, $v_{4I}$, and $v_{4I+2}$ be associated to a pair of elements on the opposite side, with domains $\Omega_{4I-1}$ and $\Omega_{4I+1}$ and normal displacement degrees-of-freedom $v_{4I-3}$, $v_{4I-1}$, and $v_{4I+1}$. The pressure variables $q_{2I-1}$, $q_{2I}$ and $q_{2I+1}$ are associated with nodes $4I-3$, $4I-2$ and $4I+1$, respectively. Nodes $4I-1$ and $4I$ are not directly associated with pressure variables. Owing to the presence of these “unassociated” nodes, the LBB condition can be satisfied in the case of the reference analytical condition for piecewise linear domain elements, as shown in Proposition F.1. A similar conclusion can be drawn for piecewise quadratic domain elements, as proved in Proposition F.2. Both propositions are contained in Appendix F.

8 A Numerical Test

The numerical solution of a simple boundary-value problem is used to test the stability and validate the error estimates for selected contact finite formulations presented earlier. The problem consists of finding the traction on the frictionless interface between two contacting and concentric cylinders subject to uniform external pressure $\bar{p} = 1$. The inner cylinder has interior radius of 10 units and exterior radius of 15 units, while the outer cylinder has interior radius of 15 units and exterior radius of 20 units. Both cylinders are assumed to be made of the same homogeneous and isotropic linearly elastic material with Young’s modulus $E = 100$ and Poisson’s ratio $\nu = 0.0$. In addition, the cylinders are taken to be infinitely long (hence, plane strain conditions apply) and the loading is quasi-static. This problem is a suitable candidate for the intended analysis for three reasons, namely: it gives rise to smooth contact over the full interface; it does not involve sliding (hence condition (9) is trivially satisfied); and it possesses an analytical solution.

Due to axisymmetry, only a slice of the cylinders needs to be analyzed subject to appropriate boundary conditions on the lateral edges. Figure 16 depicts a quarter of the domain, where the two cylinders are discretized by non-conforming meshes. Figures 17 and 18 show the rate of convergence for methods (T11MC2) and (T21MC2) under $h$-refinement. The associated finite element formulations are implemented in FEAP, a general-purpose finite element program partially documented in [23]. All integrals on the contact surface are evaluated using a method proposed in [2]. Since the contact algorithm accounts for finite deformation effects in the evaluation of integrals, the contact pressure of the exact linear elastic solution is scaled to the geometry of the deformed configuration to
render the comparison meaningful. The attained rates of convergence are approximately \(o(h)\) and \(o(h^2)\), respectively, which are higher than those predicted from (55)2. However, it should be noted that the theoretical error estimates are not necessarily sharp. In fact, the lack of sharpness can be directly attributed to the estimate of the constant \(\alpha_h\) in Proposition 3.2. This constant is shown here to be of order at least \(o(h^{1/2})\), although it is conceivable that it could, in certain cases, be of order \(o(1)\).

9 Conclusions

The stability of dual finite element formulations for the two-body contact problem can be studied analytically by means of Theorem 3.1, and Propositions 3.1 and 3.2. The reference analytical configuration proposed herein provides a test bed for the evaluation of particular contact finite elements. It is shown that, in general, formulations that employ pressure interpolations or force resultants on all boundary nodes fail the stability test. In contrast, formulations that pass the stability test generally employ pressure interpolations or force resultants that involve only a uniformly distributed subset of the boundary nodes and eliminate surface locking. A novel two-pass formulation that passes the stability test is also proposed.

Acknowledgments

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References


APPENDIX A: Fractional Sobolev spaces for two-body contact

Fractional Sobolev spaces are essential in the analysis of the two-body contact problem. A short review of the main definitions pertinent to this work is presented below. The interested reader may consult the article of Slobodeckij [24], the monograph by Lions and Magenes [25, Chapter 1], and the survey lecture by Babuška and Aziz [20] for full technical expositions.

Consider the function \( w : \Omega \subset \mathcal{E}^d \to \mathbb{R}^d \), and let \( w_i \) be its components relative to the coordinate system \( \{ e_i \} \). The norm of \( w \) in the fractional Sobolev space \( H^{n+1/2}(\Omega), \ n \in \mathbb{N} \), is defined as

\[
\| w \|_{(n+1/2)\Omega} = \left[ \| w \|_{(n,\Omega)}^2 + \sum_{i=1}^{d} \sum_{|p|=n} |D^p w_i|^2_{(1/2,\Omega)} \right]^{1/2}.
\]

Here,

\[
\| w \|_{(n,\Omega)}^2 = \sum_{i=1}^{d} \sum_{|p|\leq n} |D^p w_i|^2_{L_2(\Omega)}
\]

is the norm of the Sobolev space \( H^n(\Omega) \), where

\[
D^p = \frac{\partial^{\alpha_1+\ldots+\alpha_d}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \ , \quad p = \{ \alpha_1, \ldots, \alpha_d \} \ , \quad |p| = \alpha_1 + \ldots + \alpha_d,
\]

\( x_i \) are the components of the position vector \( x \) of a point in \( \mathbb{R}^d \), and

\[
\| w_i \|_{(1/2,\Omega)}^2 = \int_{\Omega} \int_{\Omega} \left| w_i(x) - w_i(y) \right|^2 \frac{dxdy}{\|x - y\|^{d+1}} ,
\]

where \( \| \cdot \| \) denotes the Euclidean norm in \( \mathcal{E}^d \). When \( \Omega \) is assumed Lipschitzian, a corresponding norm may be defined in the fractional Sobolev space \( H^{n+1/2}(\partial \Omega) \).

Negative Sobolev spaces \( H^{-s}(\Omega) \) and \( H^{-s}(\partial \Omega) \), \( s \in \mathbb{R}^+ \), are defined as topological duals to \( H^s(\Omega) \) and \( H^s(\partial \Omega) \), respectively. For example, the norm associated with \( H^s(\Omega) \) is

\[
\| w \|_{(-s,\Omega)} = \sup_{v \in H^s(\Omega)} \frac{\int_{\Omega} w \cdot v \, dV}{\| v \|_{(s,\Omega)}} .
\]

The following special form of a trace theorem establishes a clear connection between functions in a two-dimensional domain \( \Omega \) and associated functions on the boundary \( \Omega \), and plays an important role in the ensuing developments:
Theorem A.1 For a Lipschitzian domain \( \Omega \subset \mathcal{E}^2 \), the trace operator \( \gamma(u) = u \) maps \( H^1(\Omega) \) continuously into \( H^{1/2}(\partial \Omega) \). In addition, given any \( v \in H^{1/2}(\partial \Omega) \), there exists a \( u \in H^1(\Omega) \) such that \( \gamma(u) = v \) on \( \partial \Omega \). Furthermore, there exist constants \( c_1, c_2 > 0 \), such that

\[
c_1\|u\|_{(1,\Omega)} \leq \|v\|_{(1/2,\partial \Omega)} \leq c_2\|u\|_{(1,\Omega)}.
\]  

(A.6)

Proof: See Theorems 2.4.1 and 2.4.3 in [20, Section 2.4]. ■

A slight variation of the above theorem as applicable to domains \( \Omega \) of class \( C^{1,1} \) (i.e., whose boundary is defined by a function which is continuously differentiable with Lipschitz derivatives of order one), is recorded below:

Corollary A.1 For a \( C^{1,1} \) domain \( \Omega \subset \mathcal{E}^2 \), the normal trace operator \( \gamma_N(u) = u \cdot N \) maps \( H^1(\Omega) \) continuously into \( H^{1/2}(\partial \Omega) \). In addition, given any \( v \in H^{1/2}(\partial \Omega) \), there exists a \( u \in H^1(\Omega) \) such that \( \gamma_N(u) = v \) on \( \partial \Omega \). Furthermore, there exist constants \( c_3, c_4 > 0 \), such that

\[
c_3\|u\|_{(1,\Omega)} \leq \|v\|_{(1/2,\partial \Omega)} \leq c_4\|u\|_{(1,\Omega)}.
\]  

(A.7)

Proof: The proof follows readily from a componentwise application of Theorem A.1 and from the observations that \( vN_i \in H^{1/2}(\partial \Omega) \) if \( v \in H^{1/2}(\partial \Omega) \) and that there exists a constant \( k > 0 \), such that \( \|vN_i\|_{(1/2,\partial \Omega)} \leq k\|v\|_{(1/2,\partial \Omega)} \), where \( N_i \) denote components of the unit normal vector \( N \) to \( \partial \Omega \). ■

Two significant complications that generally arise in the mathematical analysis of the discrete two-body contact problem. Specifically, contact may occur only along a subset \( \Gamma_p \) of \( \partial \Omega \) and \( \partial \Omega \) may be non-smooth, i.e., less smooth than \( C^{1,1} \). In the former case, it is not possible to use \( H^{n-1/2}(\Gamma_p) \) in connection with Theorem A.1, since, in general, \( H^{n-1/2}(\Gamma_p) \not\subset H^{n-1/2}(\partial \Omega) \). The classical remedy for the aforementioned problem is to introduce the space \( H^{n-1/2}_{00}(\Gamma_p) \subset H^{n-1/2}(\partial \Omega) \) consisting of functions which exhibit smooth decay to zero when approaching \( \partial \Gamma_p \), see [25, Section 1.11]. Then, Theorem A.1 is obviously applicable to any \( v \in H^{n-1/2}_{00}(\Gamma_p) \).

A general treatment of contact between non-smooth domains is beyond the scope of this work. Here, the analysis of discrete contact is based upon arguments for rectangular domains \( \Omega \), where the contact boundary \( \Gamma_p \) always consists of a single straight edge. While in such a case the smoothness of \( \Gamma_p \) is obviously guaranteed, the actual distribution of normal traces precludes the use of \( H^{n-1/2}_{00}(\Gamma_p) \) even for simple problems such as the constant pressure contact patch test proposed in [22].
An important proposition applicable to contact between rectangular regions is established here immediately after the following lemma:

**Lemma A.1** Let $\Omega \in \mathcal{E}^2$ be rectangular and assume $\Gamma_p \in \partial \Omega$ is the interior of one of its edges. Also, denote by $\Gamma_p^{op} \in \partial \Omega$ the interior of the edge opposite to $\Gamma_p$, and by $\Gamma_p^{ad} = \partial \Omega / (\Gamma_p \cup \Gamma_p^{op})$ its two adjacent edges. Given a function $v \in H^{1/2}(\Gamma_p)$ which is finite at the boundary of $\Gamma_p$, there exists a $v^* \in H^{1/2}(\partial \Omega)$ such that: (i) $v^* = v$ on $\Gamma_p$, (ii) $v^* = 0$ on $\Gamma_p^{op}$, and (iii) $\|v^*\|_{(1/2, \partial \Omega)} \leq C_5 \|v\|_{(1/2, \Gamma_p)}$, where $C_5 > 0$ is a constant.

**Proof:** Let $D$ be the length of $\Gamma_p$ or, equivalently, of $\Gamma_p^{op}$, and let $L$ be the length of either of the two edges $\Gamma_p^{ad-l}$ and $\Gamma_p^{ad-r}$ of $\Omega$ comprising $\Gamma_p^{ad}$. Also, let $(x, y)$ be a rectangular Cartesian coordinate system, relative to which $\Gamma_p$ and $\Gamma_p^{op}$ have coordinates $(x, L)$ and $(x, 0)$ with $0 \leq x \leq D$, while $\Gamma_p^{ad-l}$ and $\Gamma_p^{ad-r}$ have coordinates $(0, y)$ and $(D, y)$ with $0 \leq y \leq L$. Now, consider the open subset $\mathcal{S}$ of the real line defined as

$$\mathcal{S} = \{ \xi \in \mathbb{R} \mid -D - L < \xi < D + L \}.$$  \hspace{1cm} (A.8)

The boundary $\partial \Omega$ of the rectangle can be mapped invertibly almost everywhere to $\mathcal{S}$ in a number of ways. Here, define such a mapping by means of the two coordinate charts $\theta^1 : \partial \Omega / (\partial \Gamma_p^{op} \cap \partial \Gamma_p^{ad-r}) \to \mathcal{S}$ and $\theta^2 : \partial \Omega / (\partial \Gamma_p^{op} \cap \partial \Gamma_p^{ad-l}) \to \mathcal{S}$, where

$$\theta^1(x, y) = \begin{cases} 
-x - L & \text{on } \Gamma_p^{op} \\
-L & \text{at } \Gamma_p^{op} \cap \Gamma_p^{ad-l} \\
y - L & \text{on } \Gamma_p^{ad-l} \\
0 & \text{at } \Gamma_p^{ad-l} \cap \Gamma_p \\
D & \text{at } \Gamma_p \cap \Gamma_p^{ad-r} \\
-y + D + L & \text{on } \Gamma_p^{ad-r}
\end{cases}$$ \hspace{1cm} (A.9)

and

$$\theta^2(x, y) = \begin{cases} 
y - L - D & \text{on } \Gamma_p^{ad-l} \\
-D & \text{at } \Gamma_p^{ad-l} \cap \Gamma_p \\
x - D & \text{on } \Gamma_p \\
0 & \text{at } \Gamma_p \cap \Gamma_p^{ad-r} \\
-y + L & \text{on } \Gamma_p^{ad-r} \\
L & \text{at } \Gamma_p^{ad-r} \cap \Gamma_p^{op} \\
-x + L + D & \text{on } \Gamma_p^{op}
\end{cases}.$$  \hspace{1cm} (A.10)
Then, define functions $\phi_1$ and $\phi_2$ on $\partial \Omega$ as

$$
\phi_1(x, y) = \begin{cases} 
1 & \text{on } \Gamma_p^\gamma \\
0 & \text{on } \Gamma_p^{ad} \\
\frac{1}{2} \left(1 - \cos \left(\frac{\pi y}{L_p}\right) \right) & \text{on } \Gamma_p^{ad} 
\end{cases}
$$

(A.11)

and

$$
\phi_2(x, y) = \begin{cases} 
v^*(x) & \text{on } \Gamma_p^\gamma \\
v^*(D(1 - \frac{y}{L_p})) & \text{on } \Gamma_p^{ad-l} \\
v^*(D(1 - \frac{y}{L_p})) & \text{on } \Gamma_p^{op} \\
v^*(D(\frac{L_p}{y})) & \text{on } \Gamma_p^{ad-r} 
\end{cases}
$$

(A.12)

Via the inverse mappings $(\theta^\gamma)^{-1}$ ($\gamma = 1, 2$), one can then define the functions $\hat{\phi}_1^\gamma$ and $\hat{\phi}_2^\gamma$ on $\mathcal{S}$ as

$$
\hat{\phi}_1^\gamma = \phi_1 \circ (\theta^\gamma)^{-1}, \quad \hat{\phi}_2^\gamma = \phi_2 \circ (\theta^\gamma)^{-1}.
$$

(A.13)

It is evident that $\hat{\phi}_1^\gamma$ is of class $C^1(\mathcal{S})$, therefore it is also in $H^{1/2}(\mathcal{S})$. Now, it is argued that $\hat{\phi}_2^\gamma$ also belongs to $H^{1/2}(\mathcal{S})$. To do so, one need only consider the integral

$$
\int_{-L-D}^{L+D} \int_{-L-D}^{L+D} \left( \frac{\hat{\phi}_2^\gamma(\xi) - \hat{\phi}_2^\gamma(\eta)}{\xi - \eta} \right)^2 d\xi d\eta,
$$

(A.14)

since, by assumption, $\hat{\phi}_2^\gamma$ is finite at the corners of $\Gamma_p$. This integral, in turn, can be broken up into sixteen terms corresponding to the double integrals over the combinations of all four segments of the rectangle. For brevity, let these segments be denoted by $\mathcal{S}_\zeta^\gamma$, where $\zeta \in \{1, 4\}$, and be indexed by $\zeta$ starting from the point of $\mathcal{S}$ with coordinates $\xi = -L - D$ and going clock-wise to $\xi = L + D$. Clearly, the double integral over matching segments $\mathcal{S}_\zeta^\gamma$ satisfies

$$
\int_{\mathcal{S}_\zeta^\gamma} \int_{\mathcal{S}_\zeta^\gamma} \left( \frac{\hat{\phi}_2^\gamma(\xi) - \hat{\phi}_2^\gamma(\eta)}{\xi - \eta} \right)^2 d\xi d\eta \leq C_6 |v^*|^2_{\Gamma_p^{1/2}};
$$

(A.15)

for $\gamma = 1, 2$, where $C_6$ is a positive constant. For the double integral over non-neighboring segments $\mathcal{S}_{\zeta_1}^\gamma, \mathcal{S}_{\zeta_2}^\gamma$ for which $|\zeta_2 - \zeta_1| = 2$, it is trivial to note that

$$
\int_{\mathcal{S}_{\zeta_1}^\gamma} \int_{\mathcal{S}_{\zeta_2}^\gamma} \left( \frac{\hat{\phi}_2^\gamma(\xi) - \hat{\phi}_2^\gamma(\eta)}{\xi - \eta} \right)^2 d\xi d\eta = \int_{\mathcal{S}_{\zeta_1}^\gamma} \int_{\mathcal{S}_{\zeta_1}^\gamma} \left( \frac{\hat{\phi}_2^\gamma(\xi) - \hat{\phi}_2^\gamma(\eta)}{\xi - \eta - L - D} \right)^2 d\xi d\eta
\leq C_7 |v^*|^2_{\Gamma_p^{1/2}};
$$

(A.16)
for $\gamma = 1, 2$ and $C_7$ a positive constant. For the double integral over neighboring segments, it is not hard to show that

$$\int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \left( \frac{\phi_2(\xi) - \phi_2^*(\eta)}{\xi - \eta} \right)^2 \, d\xi \, d\eta \leq C_8 \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \left( \frac{\phi_2^*(\xi) - \phi_2^*(\eta)}{\xi + \eta} \right)^2 \, d\xi \, d\eta$$

(A.17)

$$\leq C_8 |v^*|^2_{(1/2, \Gamma_p)} ,$$

for $\gamma = 1, 2$ and $C_8, C_9$ positive constants. Finally, for the double integral over non-neighboring segments for which $|\zeta_2 - \zeta_1| = 3$, one can write

$$\int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \left( \frac{\phi_2(\xi) - \phi_2^*(\eta)}{\xi - \eta} \right)^2 \, d\xi \, d\eta \leq C_9 \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \left( \frac{\phi_2^*(\xi) - \phi_2^*(\eta)}{L + D} \right)^2 \, d\xi \, d\eta$$

(A.18)

$$\leq C_9 |v^*|^2_{(1/2, \Gamma_p)} ,$$

for $\gamma = 1, 2$ and $C_9, C_9'$ positive constants. Hence, it has been shown that $\hat{\phi}_2^*$ is in $H^{1/2}(\mathcal{S})$.

Define now the function $\hat{\phi}_2^* = \hat{\phi}_1^* \hat{\phi}_2^*$ on $\mathcal{S}$. Since $\hat{\phi}_1^*$ is of class $C^1(\mathcal{S})$, one may invoke Theorem 1.4.1.1 in [26] to conclude that $\hat{\phi}_2^*$ belongs also in $H^{1/2}(\mathcal{S}^\gamma)$.

Subsequently, define the function $v = \phi_2^* \circ \theta^\gamma$ on $\partial \Omega$ and note that, with the exception of the points with coordinates $\xi = \pm (L + D)$ in $\mathcal{S}$, the function $v$ is defined independently of $\gamma$. At each of the two points where one of the mappings $\theta^1$ and $\theta^2$ does not exist, the definition of $v$ utilizes the well-defined mapping. Also note that $v$ a priori satisfies requirements (i) and (ii) of the Lemma via the definition of $\phi_1$ and $\phi_2$.

Finally, since the set $\mathcal{S}$ represents an atlas for $\partial \Omega$ through the mappings $\theta^1, \theta^2$, and via each of these mappings the quantity $v \circ \theta^{\gamma-1}$ has been shown to be bounded by $\|v^*\|_{(1/2, \Gamma_p)}$, the definition of the norm for $\partial \Omega$ in equation (1.3.3.2) of [26] implies that there exists a positive constant $C_5$ such that

$$\|v^*\|^2_{(1/2, \partial \Omega)} = \sum_{\gamma=1}^2 \|v^\gamma\|^2_{(1/2, \mathcal{S})} \leq C_5 \|v\|^2_{(1/2, \Gamma_p)} .$$

(A.19)

Therefore, the function $v$ satisfies requirement (iii), which completes the proof. \[ \qed \]

Having proved the preceding lemma, it is now possible to prove the following proposition:

\textbf{Proposition A.1} Let $\Omega \subset \mathcal{E}^2$ be rectangular and assume that $\Gamma_p \subset \partial \Omega$ is the interior of one of its edges. For every $v \in H^{1/2}(\Gamma_p)$ and finitely-valued at the boundary points of $\Gamma_p$, there exists a $u \in H^1(\Omega)$, such that its the normal trace $\gamma(u) = u \cdot \mathbf{N}$ satisfies the conditions
An analysis of dual two-body contact formulations

\( \gamma(u) = v \) on \( \Gamma_p \) and \( \gamma(u) = 0 \) on \( \partial \Omega / \Gamma_p \). In addition, there exist positive constants \( c_5 \) and \( \alpha_6 \), such that

\[
c_5 \| u \|_{(1, \Omega)} \leq \| v \|_{(1, \Omega)} \leq c_6 \| u \|_{(1, \Omega)}. \tag{A.20}
\]

**Proof:** Appealing to the results of Lemma A.1 and using its nomenclature, one may construct a function \( v^* \in H^{1/2}(\partial \Omega) \), such that \( v^* = v \) on \( \Gamma_p \), \( v^* = 0 \) on \( \Gamma_p^0 \), and

\[
\| v \|_{(1/2, \Gamma_p)} \leq \| v^* \|_{(1/2, \partial \Omega)} \leq C_5 \| v \|_{(1/2, \Gamma_p)}, \tag{A.21}
\]

where \( C_5 > 0 \). Since \( \Omega \) is Lipschitzian, Theorem A.1 implies that there exists a function \( u^* \in H^1(\Omega) \) and positive constants \( c_1^* \) and \( c_2^* \), such that \( u^* = v^* \) on \( \partial \Omega \) and

\[
c_1^* \| u^* \|_{(1, \Omega)} \leq \| v^* \|_{(1/2, \partial \Omega)} \leq c_2^* \| u^* \|_{(1, \Omega)}. \tag{A.22}
\]

Now, let \( u = u^* N_c \), where \( N_p \) is the outward unit normal to \( \Gamma_p \). It follows that

\[
\| u \|_{(1, \Omega)} = \| u^* \|_{(1, \Omega)} \tag{A.23}
\]

and

\[
u \cdot N = \begin{cases} 
  u^* N_c \cdot N_p = v^* = v & \text{on } \Gamma_c \\
  u^* N_c \cdot N_p^{ad} = 0 & \text{on } \Gamma_c^{ad} \\
  u^* N_c \cdot N_p^{op} = -v^* = 0 & \text{on } \Gamma_c^{op}
\end{cases} \tag{A.24}
\]

Combining (A.21) and (A.22) with (A.23), it follows that the inequalities in (A.20) hold true for \( c_5 = c_1^*/C_5 \) and \( c_6 = c_2^* \). ■

**APPENDIX B: Lemmas and theorems on the \( \| \cdot \|_{(1/2, \Gamma)} \) norm**

A series of preliminary technical results is recorded below.

**Lemma B.1** Given \( \Gamma = (0, 1) \), the inequality

\[
|x^n|_{(1/2, \Gamma)} \geq 1, \quad \forall \ n \in \mathbb{N} \tag{B.1}
\]

holds true.

**Proof:** Recalling (A.1), it follows that

\[
|x^n|_{(1/2, \Gamma)}^2 = \int_0^1 \int_0^1 \left( \frac{x^n - y^n}{x - y} \right)^2 \ dx \ dy \tag{B.2}
\]
The right-hand side of (B.2) can be written as

\[
\int_0^1 \int_0^1 \left( \frac{x^n - y^n}{x - y} \right)^2 \, dx \, dy = \int_0^1 \int_0^1 \left( \sum_{i=1}^n x^{n-i} y^{j-1} \right)^2 \, dx \, dy \\
= \sum_{i=1}^n \sum_{j=1}^n \int_0^1 \int_0^1 x^{2n-i-j} y^{i+j-2} \, dx \, dy \\
= \sum_{i=1}^n \sum_{j=1}^n \frac{1}{(2n - i - j + 1)(i + j - 1)}. \tag{B.3}
\]

Since it is clear that

\[
\max_{i,j \in \{1, \ldots, n\}} (2n - i - j + 1)(i + j - 1) = n^2, \tag{B.4}
\]

one may invoke equations (B.2) and (B.3) to conclude that

\[
|x^n|_{(1/2, \Gamma)} \geq \sum_{i=1}^n \sum_{j=1}^n \frac{1}{n^2} = 1. \tag{B.5}
\]

**Lemma B.2** Let \(P : \mathbb{R} \mapsto \mathbb{R}\) be a polynomial function, such that \(P(0) = 0\) and \(P(1) = 1\). Then, given \(\Gamma = (0, 1)\), it follows that \(|P(x)|_{(1/2, \Gamma)} \geq 1\).

**Proof:** The polynomial can be expressed as \(P(x) = \sum_{i=1}^n a_i x^i\), where \(\sum_{i=1}^n a_i = 1\). Equation (A.1) again yields

\[
|P(x)|^2_{(1/2, \Gamma)} = \int_0^1 \int_0^1 \left( \frac{\sum_{i=1}^n a_i x^i - \sum_{j=1}^n a_j y^j}{x - y} \right)^2 \, dx \, dy, \\
= \int_0^1 \int_0^1 \sum_{i=1}^n \sum_{j=1}^n a_i a_j \frac{(x^i - y^i)(x^j - y^j)}{(x - y)^2} \, dx \, dy, \tag{B.6}
\]

For each pair \((i, j)\) in (B.6)_3, note that

\[
\frac{x^i - y^i}{x - y} \frac{x^j - y^j}{x - y} > 0, \tag{B.7}
\]

hence, also,

\[
\frac{x^i - y^i}{x - y} \frac{x^j - y^j}{x - y} \geq \min_{k=i,j} \left( \frac{x^k - y^k}{x - y} \right)^2. \tag{B.8}
\]

Invoking Lemma B.1 and equation (B.8), it follows from (B.6)_3 that

\[
|P(x)|^2_{(1/2, \Gamma)} \geq \sum_{i=1}^n \sum_{j=1}^n a_i a_j = 1. \tag{B.9}
\]
Lemma B.3 Let $P : \mathbb{R} \to \mathbb{R}$ be a polynomial function, such that $P(0) = 0$ and $P(L) = 1$, where $L > 0$. Then, given $\Gamma = (0, L)$, it follows that $|P(x)|_{(1/2, \Gamma)} \geq 1$.

Proof: Write $P(x) = \sum_{i=1}^{n} a_i(x - \frac{x}{L})^i$, with $\sum_{i=1}^{n} a_i = 1$, let $\xi = \frac{x}{L}$, $\eta = \frac{\xi}{L}$, and invoke Lemma B.2.

Lemma B.4 Given $\Gamma = (0, 1)$, let $f : \Gamma \to \mathbb{R}$ belong to $H^{1/2}(\Gamma)$, such that $f(0) = 0$ and $f(1) = 1$. Then $\|f(x)\|_{(1/2, \Gamma)} \geq 1 - \epsilon$ for any positive $\epsilon$.

Proof: Since $C^\infty$ is dense in $H^{1/2}(\Gamma)$, see [26, Theorem 1.4.2.1], given any $f \in H^{1/2}(\Gamma)$ and $\epsilon_1 > 0$, choose $\phi \in C^\infty(\Gamma)$, such that

$$\|f - \phi\|_{(1/2, \Gamma)} \leq \epsilon_1.$$  \hfill (B.10)

Subsequently, appealing to the Weirstrass polynomial approximation theorem, choose a polynomial $P$ and a constant $\epsilon_2 > 0$, such that

$$\left\| \frac{dP}{dx} - \frac{d\phi}{dx} \right\|_{(\infty, \Gamma)} \leq \epsilon_2.$$  \hfill (B.11)

This, in turn, implies that $P$ can be chosen to satisfy

$$\|P - \phi\|_{(0, \Gamma)}^2 \leq \epsilon_2^2.$$  \hfill (B.12)

Now, let $e = P - \phi$ and write

$$|e|_{(1/2, \Gamma)}^2 = \int_0^1 \int_0^1 \frac{|e(x) - e(y)|^2}{(x-y)^2} \, dx \, dy.$$  \hfill (B.13)

By the standard intermediate value theorem

$$e(x) - e(y) = \frac{de}{dx}(z)(x - y),$$  \hfill (B.14)

for some $z \in (x, y)$, hence, upon recalling (B.11),

$$|e|_{(1/2, \Gamma)}^2 = \int_0^1 \int_0^1 \left[ \frac{de(z(x,y))}{dx} \right]^2 \, dx \, dy \leq \int_0^1 \int_0^1 \epsilon_2^2 = \epsilon_2^2.$$  \hfill (B.15)

It follows from (B.12) and (B.15) that

$$\|e\|_{(1/2, \Gamma)}^2 = \|e\|_{(0, \Gamma)}^2 + |e|_{(1/2, \Gamma)}^2 \leq 2\epsilon_2^2.$$  \hfill (B.16)

Taking into account (B.10) and (B.16), and letting $\epsilon_1 = \epsilon_2 = \frac{\epsilon}{\sqrt{3}}$, it follows that

$$\|f - P\|_{(1/2, \Gamma)}^2 \leq \|f - \phi\|_{(1/2, \Gamma)}^2 + \|\phi - P\|_{(1/2, \Gamma)}^2 \leq \epsilon_1^2 + 2\epsilon_2^2 = \epsilon^2,$$  \hfill (B.17)
which shows that the Weirstrass approximation property of polynomials holds also true in \( H^{1/2}(\Gamma) \). Now, \( P \) can be chosen such that \( P(x) = 0 \) and \( P(1) = 1 \). Then, by the triangle inequality and Lemma B.2 it is concluded that

\[
\| f(x) \|_{(1/2, \Gamma)} \geq \| P(x) \|_{(1/2, \Gamma)} - \epsilon \geq 1 - \epsilon ,
\]

for any positive \( \epsilon \). ■

**Proposition B.1** Let \( \Gamma = (0, 1) \) and \( f : \Gamma \mapsto \mathbb{R} \) a \( C^0 \) piecewise polynomial such that \( |f(0)| = |f(1)| = 1 \). Then for any \( \epsilon > 0 \), \( \| f(x) \|_{1/2, \Gamma} > (\frac{2}{3} - \epsilon)^{1/2} \).

**Proof:** Consider first the case \( f(0) = f(1) = 1 \). Define \( d = \min_{\Gamma} f(x) \) and let \( \tilde{x} \) be such that \( f(\tilde{x}) = d \), where it is understood that \( \tilde{x} \) may not be unique. The proposition holds trivially true if \( d \geq 1 \). Therefore, assume that \( d < 1 \), define the coordinates \( \xi_1 = \frac{x - \tilde{x}}{\tilde{x}} \) and \( \xi_2 = \frac{x - \tilde{x}}{1 - \tilde{x}} \), and consider the functions \( f^*(\xi_1) = \frac{1}{1 - d} [f((1 - \xi_1)\tilde{x}) - d] \) and \( g^*(\xi_2) = \frac{1}{1 - d} [f((1 - \xi_2)\tilde{x} + \tilde{x}) - d] \), such that \( f^*(0) = g^*(0) = 0 \), \( f^*(1) = g^*(1) = 1 \). Letting \( \tilde{d} = \max(d, 0) \), it follows that

\[
\| f(x) \|_{(1/2, \Gamma)}^2 \geq \int^1_0 \int^1_0 [f(x) - f(y)]^2 \frac{dx dy}{(x - y)^2} + \int^1_\tilde{x} \int^1_{\tilde{x}} [f(x) - f(y)]^2 \frac{dx dy}{(x - y)^2} + \int^1_\tilde{x} \int^1_{\tilde{x}} [f(\xi_1) - f(\eta_1)]^2 \frac{d\xi_1 d\eta_1}{(\xi_1 - \eta_1)^2}
\]

\[
+ \int^1_0 \int^1_0 [g^*(\xi_2) - g^*(\eta_2)]^2 \frac{d\xi_2 d\eta_2}{(\xi_2 - \eta_2)^2}.
\]

(B.19)

Invoking Lemmas B.3 and B.4, it follows from (B.19) that

\[
\| f(x) \|_{(1/2, \Gamma)}^2 \geq d^2 + 2(1 - d)^2 - \epsilon \geq \frac{2}{3} - \epsilon .
\]

(B.20)

Now, consider the case \( f(0) = -1 \) and \( f(1) = 1 \). First, let \( \bar{x} \) be such that \( f(\bar{x}) = 0 \) and note, again, that \( \bar{x} \) need not be unique. Subsequently, define the coordinates \( \xi_1 \) and \( \xi_2 \) as earlier and set \( f^*(\xi_1) = -f((1 - \xi_1)\bar{x}) \) and \( g^*(\xi_2) = f((1 - \xi_2)\bar{x}) \), corresponding to the previous definitions for \( d = 0 \). Again, it is clear that \( f^*(0) = g^*(0) = 0 \) and \( f^*(1) = g^*(1) = 1 \). It follows that

\[
\| f(x) \|_{(1/2, \Gamma)}^2 \geq \int^\bar{x} \int^\bar{x} [f(x) - f(y)]^2 \frac{dx dy}{(x - y)^2} + \int^1_{\bar{x}} \int^1_{\bar{x}} [f(x) - f(y)]^2 \frac{dx dy}{(x - y)^2} + \int^1_{\bar{x}} \int^1_{\bar{x}} [f(\xi_1) - f(\eta_1)]^2 \frac{d\xi_1 d\eta_1}{(\xi_1 - \eta_1)^2}
\]

\[
+ \int^1_0 \int^1_0 [g^*(\xi_2) - g^*(\eta_2)]^2 \frac{d\xi_2 d\eta_2}{(\xi_2 - \eta_2)^2}.
\]

(B.21)
so that appealing to Lemmas B.3 and B.4 it is concluded that
\[
\|f(x)\|_{(1/2, 1)}^2 \geq 2 - \epsilon ,
\]
for arbitrary \(\epsilon > 0\). The proof for the cases \(f(0) = 1\), \(f(1) = -1\) and \(f(0) = f(1) = -1\) follow similarly.

**APPENDIX C: Inverse Assumption for Constant-size Discretizations of 1-D Domains using Fractional Norms**

Preliminary to establishing the relevant inverse assumption for one-dimensional domains, consider the following lemmas:

**Lemma C.1** For any \(a, \epsilon\) and \(r\) that satisfy \(|a| \geq 2, \epsilon > 0,\) and \(0 \leq r < 1 - \epsilon\), the inequality

\[
\int_0^1 \int_0^1 \frac{(u(\xi) - v(\eta))^2}{|a + \xi - \eta|^{1+2r}} \, d\xi \, d\eta \geq \int_0^1 \int_0^1 \frac{(u(\xi) - v(\eta))^2}{|a + \xi - \eta|^{3-2\epsilon}} \, d\xi \, d\eta ,
\]

holds true for all functions \(u\) and \(v\) on \((0, 1)\) for which the above integrals exist.

**Proof:** For any \(\xi\) and \(\eta\) in \((0, 1)\) and for any \(a\) with \(|a| \geq 2\), one has

\[
|a + \xi - \eta| \geq 1 .
\]

Hence, for \(r < 1 - \epsilon\),
\[
|a + \xi - \eta|^{1+2r} \leq |a + \xi - \eta|^{3-2\epsilon} ,
\]

therefore
\[
\frac{(u(\xi) - v(\eta))^2}{|a + \xi - \eta|^{1+2r}} \geq \frac{(u(\xi) - v(\eta))^2}{|a + \xi - \eta|^{3-2\epsilon}} ,
\]

which proves the lemma. 

**Lemma C.2** Let \(u \in \mathcal{U}\), where \(\mathcal{U}\) is a finite-dimensional polynomial subspace of \(H^r(0, 1)\) with \(r \geq 0\), and recall that
\[
|u|_{(r, (0,1))}^2 = \int_0^1 \int_0^1 \frac{(u(\xi) - u(\eta))^2}{|\xi - \eta|^{1+2r}} \, d\xi \, d\eta .
\]

Then, given any \(\epsilon > 0\), where \(r < 1 - \epsilon\), there exists a \(C_{10} > 0\) independent of \(u\), such that
\[
|u|_{(r, (0,1))} \geq C_{10}|u|_{(1-\epsilon, (0,1))} .
\]
Proof: Consider first all functions \( u \in \mathcal{U} \) which satisfy \( u(0) = 0 \). Referring to this subspace of \( \mathcal{U} \) as \( \mathcal{U}_0 \), note that \( |u_{|_{C_{10}}} \) is clearly a norm for \( \mathcal{U}_0 \). Then, given that \( \mathcal{U} \) is finite-dimensional, there exists a \( C_{10} > 0 \), such that

\[
|u_{|_{C_{10}}} \geq C_{10} |u_{(1-\epsilon,0,1)}| , \tag{C.5}
\]

for all \( u \in \mathcal{U}_0 \). Since \( |u_{|_{C_{10}}} \leq |u_{|_{C_{0}}} \) when \( r \leq s \), choose \( C_{10} \) to correspond to \( r = 0 \), in which case equation (C.5) is valid for any positive \( r < 1 - \epsilon \). In case \( u(0) \neq 0 \), define \( v = u - c \) on \( (0,1) \), where \( v(0) = 0 \) and \( c \) is a constant. It follows from (C.3) and the previous argument that

\[
|v_{|_{C_{10}}} \geq C_{10} |v_{(1-\epsilon,0,1)}| = C_{10} |v_{|(1-\epsilon,0,1)}| . \tag{C.6}
\]

Lemma C.3 Let \( u \in \mathcal{U} \), where \( \mathcal{U} \) is a continuous finite-dimensional piecewise polynomial subspace of \( H^r(0,2) \) with \( 0 \leq r \), and define

\[
<u>^2_{(r,0,2)} = \int_0^1 \int_1^2 \frac{(u(\xi) - u(\eta))^2}{|\xi - \eta|^{1+2r}} \, d\xi \, d\eta . \tag{C.7}
\]

Then, there exists a constant \( C_{11} > 0 \) independent of \( r \), such that

\[
<u>^2_{(r,0,2)} \geq C_{11} <u>_{(1-\epsilon,0,2)} . \tag{C.8}
\]

Proof: It is established first that \( <u>_{(r,0,2)} \) is a semi-norm for \( \mathcal{U} \). To this end, define \( w(\xi,\eta) \) on \((0,1) \times (1,2)\) as

\[
w(\xi,\eta) = \frac{u(\xi) - u(\eta)}{|\xi - \eta|^{1/2+r}} , \tag{C.9}
\]

and note that

\[
<u>^2_{(r,0,2)} = \int_0^1 \int_1^2 w^2(\xi,\eta) \, d\xi \, d\eta = \|w\|^2_{(0,1) \times (1,2)} . \tag{C.10}
\]

Therefore,

\[
\|w_1 + w_2\|_{(0,1) \times (1,2)} \leq \|w_1\|_{(0,1) \times (1,2)} + \|w_2\|_{(0,1) \times (1,2)} \tag{C.11}
\]

for any two functions \( w_1 \) and \( w_2 \) in \( H^0((0,1) \times (1,2)) \). However, since

\[
w_1 + w_2 = \frac{(u_1(\xi) + u_2(\xi)) - (u_1(\eta) - u_2(\eta))}{|\xi - \eta|^{1/2+r}} , \tag{C.12}
\]

the triangular inequality

\[
<u_1 + u_2>_{(r,0,2)} \leq <u_1>_{(r,0,2)} + <u_2>_{(r,0,2)} \tag{C.13}
\]
holds true for all \( u_1 \) and \( u_2 \) in \( \mathcal{U} \). Likewise, the other semi-norm properties are easily verified.

Now, consider a function \( u \in \mathcal{U} \) for which \( < u >_{(r,0,2)} = 0 \). Recalling (C.7) and the fact that \( u \) is continuous, one must have that

\[
    u(\xi) - u(\eta) = 0 ,
\]

for all \((\xi, \eta) \in (0, 1) \times (1, 2)\). Hence, the function \( u \) must be constant. This implies that \(< \cdot >\) is a norm for the subspace \( \mathcal{U}_0 \) of \( \mathcal{U} \) containing functions that vanish identically at zero. Given that \( \mathcal{U}_0 \) is finite-dimensional, there exists a constant \( C_{11} > 0 \) independent of \( u \) such that

\[
    < u >_{(r,0,2)} \geq C_{11} < u >_{(1-\epsilon,0,2)} .
\]

In case \( u(0) \neq 0 \), write \( u = v + c \) where \( v(0) = 0 \) and \( c \) is a constant. Then

\[
    < u >_{(r,0,2)} = < v >_{(r,0,2)} \geq C_{11} < v >_{(1-\epsilon,0,2)} = C_{11} < u >_{(1-\epsilon,0,2)} .
\]

In either case, the constant \( C_{11} \) is independent of \( u \), but dependent on \( r \). Since one is concerned only with values of \( r \) in the interval \([0, 1-\epsilon)\) and it is trivially true from (C.15) that \( C_{11} = 1 \) for \( r = 1-\epsilon \), it follows that \( C_{11} \) can be taken to be the minimum over the closed interval \([0, 1-\epsilon]\). Hence, \( C_{11} \) is independent of \( r \).

**Lemma C.4** Let \( \Gamma \) be a one-dimensional domain of length \( L \), discretized by a sequence of \( C^0 \) piecewise polynomial discretizations parameterized by \( N \), where for each discretization there are \( N \) piecewise polynomial finite elements \( \Omega^I \) of length \( h = \frac{L}{N} \), and recall that

\[
|u|^2_{(r, \Gamma)} = \int_{\Gamma} \int_{\Gamma} \frac{(u(x) - u(y))^2}{|x-y|^{1+2r}} \, dx \, dy .
\]

Then, given any \( \epsilon \in (0, 1) \) and positive \( r < 1-\epsilon \), there exists a constant \( \bar{C} > 0 \) independent of \( r \) such that

\[
|u|_{(r, \Gamma)} \geq \bar{C} h^{-(r+\epsilon-1)} |u|_{(1-\epsilon, \Gamma)} .
\]

**Proof:** First, write

\[
|u|^2_{(r, \Gamma)} = \sum_{I=1}^{N} \sum_{J=1}^{N} \int_{\Omega^I} \int_{\Omega^J} \frac{(u(x) - u(y))^2}{|x-y|^{1+2r}} \, dx \, dy .
\]

Within each element domain \( \Omega^I \), let

\[
x = h((I-1) + \xi) , \quad u^I(\xi) = u(h((I-1) + \xi)).
\]
Then, express the integral of equation (C.19) as
\[ |u|^2_{(r,1)} = h^{1-2r} \sum_{I=1}^{N} \sum_{J=1}^{N} \int_0^1 \int_0^1 \frac{(u^I(\xi) - u^J(\eta))^2}{|I - J + \xi - \eta|^{1+2r}} \, d\xi \, d\eta . \] 
(C.21)

Now, define
\[ I^{IJ}_r = \int_0^1 \int_0^1 \frac{(u^I(\xi) - u^J(\eta))^2}{|I - J + \xi - \eta|^{1+2r}} \, d\xi \, d\eta , \] 
(C.22)

so that
\[ |u|^2 = h^{1-2r} \sum_{I=1}^{N} \sum_{J=1}^{N} \left( I^{IJ}_r + I^{(I-1)}_r + I^{I(I-1)}_r + \sum_{J=I+2}^{N} [I^{IJ}_r] \right) . \] 
(C.23)

By Lemma C.1, given any \( \epsilon \in (0, 1) \) such that \( 0 \geq r < 1 - \epsilon \), one has that
\[ I^{IJ}_r \geq I^{IJ}_{1-\epsilon} \] 
(C.24)

for \( |I - J| \geq 2 \). Likewise, by virtue of Lemma C.2 it follows that
\[ I^{II}_r = |u|^2_{(r,(0,1))} \geq C_{10}^2 |u|^2_{(1-\epsilon,(0,1))} = C_{10}^2 I^{II}_{1-\epsilon} , \] 
(C.25)

where \( C_{10} \) is a positive constant independent of \( r \). For the remaining terms \( I^{(I-1)}_r \) and \( I^{I(I-1)}_r \) in (C.23), write
\[ I^{(I-1)}_r = \int_0^1 \int_0^2 \frac{(\bar{u}^I(\xi) - \bar{u}^I(\eta))^2}{|\xi - \eta|^{1+2r}} \, d\xi \, d\eta = < \bar{u}^2 >_{(r,(0,2))} , \] 
(C.26)

where
\[ \bar{u}^I(\xi) = \begin{cases} u^I(\xi - 1) & \text{for } 1 \leq \xi \leq 2 \\ u^{I-1}(\xi) & \text{for } 0 \leq \xi < 1 \end{cases} . \] 
(C.27)

and \( < \bar{u} >_{(r,(0,2))} \) is defined as in Lemma C.3. The same lemma implies that there exists a positive constant \( C_{11} \) independent of \( r \) such that
\[ < \bar{u} >_{(r,(0,2))} \geq C_{11} < \bar{u} >_{(1-\epsilon,(0,2))} , \] 
(C.28)

hence,
\[ I^{(I-1)}_r \geq C_{11} I^{(I-1)}_{1-\epsilon} . \] 
(C.29)

Following an analogous procedure, it is readily established that
\[ I^{I(I+1)}_r \geq C_{11} I^{I(I+1)}_{1-\epsilon} . \] 
(C.30)
An analysis of dual two-body contact formulations

With the aid of (C.24), (C.25), (C.29) and (C.30), equation (C.23) implies that

\[ |u|^2_{(r,\Gamma)} \geq h^{-2r} \sum_{I=1}^{I-2} \left( \sum_{J=1}^{J_{I-1}} I_{I_{I-1}}^J + C_{10}^2 I_{1-1}^J + C_{11}^2 I_{1-1}^J + C_{11}^2 I_{1-1}^{J+1} + \sum_{J=I+1}^{N} I_{I_{I-1}}^J \right), \]

\[ \geq \min(C_{10}^2, C_{11}^2, 1) h^{-2(r+\epsilon-1)} |u|^2_{(1-\epsilon,\Gamma)}, \]

where use is also made of (C.17). Then, the proof follows upon taking \( C = \left( \max(C_{10}^2, C_{11}^2, 1) \right)^{1/2}. \)

The main result in this appendix follows:

**Theorem C.1 (inverse assumption)** Let \( \Gamma \) be a one-dimensional domain of length \( L \), discretized by a sequence of \( C^0 \) piecewise polynomials parametrized by \( N \), where for each discretization there are \( N \) piecewise polynomial finite elements \( \Omega^j \) of length \( h = \frac{L}{N} \). Given any \( \epsilon \in (0,1) \), there exists a positive constant \( \tilde{C} \) independent of \( u \), such that for small enough \( h \),

\[ \| u \|_{(s_2,\Gamma)} \leq \tilde{C} h^{s_1-s_2} \| u \|_{(s_1,\Gamma)}, \]

where \( 0 \leq s_1 \leq s_2 \leq 1 - \epsilon. \)

**Proof:** Consider any positive \( r < 1 - \epsilon \) and recall that

\[ |u|^2_{(r,\Gamma)} = |u|^2_{(0,\Gamma)} + |u|^2_{(r,\Gamma)}. \]

(C.33)

Using Lemma C.4, one may write

\[ |u|^2_{(r,\Gamma)} \geq \min(\tilde{C}^2 h^{-2(r+\epsilon-1)}, 1) |u|^2_{(1-\epsilon,\Gamma)}, \]

(C.34)

where \( \tilde{C} \) is a positive constant independent of \( r \), which, it turn, readily implies that

\[ |u|^2_{(r,\Gamma)} \geq \tilde{C} h^{-(r+\epsilon-1)} |u|^2_{(1-\epsilon,\Gamma)}. \]

(C.35)

Then, the proof follows by appealing to [20, Theorem 4.1.3].

**APPENDIX D: Babuška-Brezzi condition for node-on-surface methods**

Proposition 6.1 can be employed to analyze various node-on-surface formulations. Here, attention is focused exclusively on two-pass versions due to their widespread use, their geometric uniaxial, and their ability, in certain cases, to perform the contact patch test.
Proposition D.1 Any pressure interpolation that corresponds to a two-pass node-on-surface formulation employing a domain displacement interpolation based on 3-node triangles or 4-node quadrilaterals (method T1N2) fails condition (34) of Theorem 3.1.

Proof: With reference to Figure 8, let the magnitude of the resultant force corresponding to the pressure field \( \tilde{p} \) satisfy

\[
\tilde{Q}_{2I-1} = -\tilde{Q}_{2I} = Q , \; \forall I = 1, 2, \ldots ,
\]

where \( Q \) is a non-zero constant. Then, for any node \( K \) on \( \Gamma_p^\alpha \) sufficiently removed from the edge of contact,

\[
\sum_{J=1}^{M} \tilde{Q}_J \Phi_K^\alpha (s_J^\alpha) N^\alpha (s_J^\alpha) = \left[ Q - (1 - \beta)Q - \beta Q \right] N^\alpha = 0 .
\]

Hence, choosing \( \tilde{Q}_J \) as above and taking any \( \nu_I^\alpha \) with non-zero value only at nodes sufficiently removed from the edge of contact, one has

\[
\sum_{J=1}^{M} \left[ \tilde{Q}_J \sum_{\alpha=1}^{N^\alpha} \sum_{I=1}^{2} \left( \Phi_J^\alpha (s_J^\alpha) N^\alpha (s_J^\alpha) \right) \cdot \nu_I^\alpha \right] = 0 .
\]

Therefore, by Proposition 6.1, method T1N2 fails condition (34) of Theorem 3.1.

At this stage, it is illuminating to deduce a pressure interpolation that corresponds to the preceding node-on-surface scheme, in the sense that it yields identical discrete constraints and equivalent nodal forces to those of the latter. To this end, given displacement and pressure interpolations as in Section 6.2, as well as an integration rule on \( \Gamma_p^\alpha \), express the discrete counterparts of equations (18) as

\[
\sum_{J=1}^{N^\alpha} A_{IJ}^\alpha u_J^\alpha + \sum_{K=1}^{M} B_{KI}^\alpha p_K = F_I^\alpha , \; \forall I
\]

\[
\sum_{\alpha=1}^{2} \left[ (q_K - p_K) \sum_{J=1}^{N^\alpha} B_{KI}^\alpha \cdot (X_J^\alpha + u_J^\alpha) \right] \geq 0 , \; \forall K .
\]

In the preceding equations, the arrays \( A_{IJ}^\alpha \) and \( B_{KI}^\alpha \) are defined by

\[
u_I^\alpha \cdot (A_{IJ}^\alpha \nu_I^\alpha) = A^\alpha (\Phi_J^\alpha u_J^\alpha, \Phi_J^\alpha \nu_I^\alpha) ,
\]

\[
B_{KI}^\alpha \cdot \nu_I^\alpha = B^\alpha (\Phi_K^\alpha, \Phi_J^\alpha \nu_I^\alpha) , \quad \text{(D.5)}
\]

\[
F_I^\alpha \cdot \nu_I^\alpha = f^\alpha (\Phi_I^\alpha \nu_I^\alpha) .
\]
The effect of numerical integration on the contact interface can be readily included by replacing $\mathbf{B}^\alpha_{KI}$ in (D.5) with an approximation $\tilde{\mathbf{B}}^\alpha_{KI}$. In a potentially corresponding node-one-surface method, the discrete problem reads

\[
\sum_{J=1}^{N^a} A^\alpha_{IJ} \mathbf{u}^\alpha_J + \sum_{K=1}^M \mathbf{B}^\alpha_{KI} P_K = \mathbf{F}^\alpha_I, \quad \forall I
\]

\[
\sum_{\alpha=1}^2 \left[ (Q_K - P_K) \sum_{J=1}^{N^a} \mathbf{B}^\alpha_{KI} \cdot (\mathbf{X}^\alpha_J + \mathbf{u}^\alpha_J) \right] \geq 0, \quad \forall K,
\]

where the bilinear form $\mathbf{B}^\alpha_{KI}$ emanates from the node-on-surface approximation with pressure resultants $P_K$ and $Q_K$.

A correspondence between two formulations that employ pressure interpolation and node-on-surface approximation is established if there exists an invertible transformation matrix $[C_{KL}]$ relating the resultants $Q_K$ to the pressures $q_L$ by

\[
Q_K = \sum_{L=1}^M C_{KL} q_L,
\]

such that

\[
\tilde{\mathbf{B}}^\alpha_{KI} = \sum_{L=1}^M C_{LK} \tilde{\mathbf{B}}^\alpha_{LI}, \quad \forall K, I.
\]

Equation (D.8) implies that the two formulations are equivalent in the sense that pressures and resultants related by (D.7) yield the same displacement solution when applied to problems (D.4) and (D.6), respectively. Clearly, the aforementioned equivalence depends crucially on the choice of numerical integration that yields $\tilde{\mathbf{B}}^\alpha_{IJ}$.

**Proposition D.2** The formulations T11C2 and T1N2 are equivalent, when the contact boundary integrals in the former are evaluated using the piecewise-trapezoidal rule over integration segments with endpoints defined by adjacent nodes.

**Proof:** Start with formulation T11C2 and evaluate $\tilde{\mathbf{B}}^\alpha_{KI}$ using the piecewise-trapezoidal rule over integration segments with endpoints defined by adjacent nodes. This, with reference to Figure 7, implies that a typical integration cell would have nodes $2I - 1$ and $2I$ as its endpoints. At any endpoint $K$ of a cell, only the pressure interpolation function $\Psi_K^\alpha$ is non-zero. Letting $w_K$ be the integration weight at $K$, one can write

\[
\tilde{\mathbf{B}}^\alpha_{KI} = -w_K \Psi_K^\alpha(s_K^\alpha) \mathbf{N}^\alpha(s_K^\alpha).
\]
On the other hand, the node-on-surface formulation yields

\[
\hat{B}^\alpha_{KI} = -\Phi_I^\alpha(s_K^\alpha) N^\alpha(s_K^\alpha),
\]  

(D.10)
hence \(C_{KL} = w_K \delta_{KL}\), where \([\delta_{KL}]\) is the \(M \times M\) identity matrix.

With the use of the preceding propositions, it is simple to prove the following result:

**Corollary D.1** The two-pass linear pressure interpolation for two-body contact using linear domain elements (method T11C2) fails the LBB condition when the contact boundary integrals are evaluated using the piecewise-trapezoidal rule over integration segments with endpoints defined by adjacent nodes.

**Proof:** By Proposition D.2, methods T11C2 and T1N2 are equivalent given the specified integration rule. Then, by Proposition D.1, method T1N2 and its equivalent pressure interpolation methods fail the LBB condition. ■

The corresponding result to Proposition D.1 for piecewise quadratic domain elements follows:

**Proposition D.3** Any pressure interpolation that corresponds to a two-pass node-on-surface formulation employing a domain displacement interpolation based on 6-node triangles or 8/9-node quadrilaterals (method T2N2) fails condition (34) of Theorem 3.1.

**Proof:** Referring to Figure 12, let a representative edge node on the lower surface be numbered \(4I\), an edge node on the upper surface be numbered \(4I - 3\), and, likewise, a center node on the lower and upper surface be numbered \(4I - 2\) and \(4I - 1\), respectively.

Choose the magnitude \(\tilde{Q}_J\) of the resultant force at node \(J\) to satisfy

\[
\begin{align*}
\tilde{Q}_{4I-3} &= -\tilde{Q}_{4I} = Q_1, \\
\tilde{Q}_{4I-2} &= -\tilde{Q}_{4I-1} = Q_2,
\end{align*}
\]  

(D.11)

where \(Q_1\) and \(Q_2\) are non-zero constants. As depicted in the figure, let the nodes of the top surface (say, \(\alpha = 2\)) be offset by a distance \(\beta h\) \((0 < \beta < 1)\) relative to the bottom surface (say, \(\alpha = 1\)), where \(h\) is the (constant) surface node spacing. Now, calculate the equivalent nodal force due to contact for a center node \(4I - 1\) on the top surface as

\[
\sum_{K=1}^{M} \tilde{Q}_K \hat{B}^2_{K(4I-1)} = -[Q_2(1 - \beta)^2 - Q_1(1 - \beta^2)] N^2. 
\]  

(D.12)
An identical result is obtained for a center node on the bottom surface, to within a sign change. Subsequently, consider an edge node $4I - 4$ on the bottom surface, for which

$$
\sum_{K=1}^{M} \bar{Q}_K \hat{B}^1_K(4I-4) = -[-Q_1(1 - \beta^2) + Q_2(1 - \beta)^2]N^1.
$$

(D.13)

Hence, upon choosing $\bar{Q}_J$ such that

$$
Q_2 = \frac{1 + \beta}{1 - \beta} Q_1,
$$

(D.14)

and considering any $\mathbf{v}^I_j$ with non-zero values only at nodes sufficiently removed from the edge of contact, it follows that

$$
\sum_{J=1}^{M} \bar{Q}_J \Phi^1_K(s^1_J)N^1(s^1_J) = \sum_{J=1}^{M} \bar{Q}_J \Phi^2_K(s^2_J)N^2(s^2_J) = 0.
$$

(D.15)

Hence, for such $\mathbf{v}^I_j$, one has

$$
\sum_{J=1}^{M} \left[ \bar{Q}_J \sum_{a=1}^{2} \sum_{I=1}^{N^a} \left( \Phi^a_I(s^a_J)N^a(s^a_J) \right) \cdot \mathbf{v}^I_j \right] = 0.
$$

(D.16)

Therefore, by Proposition 6.1, method T2N2 fails condition (34) of Theorem 3.1. ■

Note that the preceding analysis does not apply when $\beta = 0$ and $\beta = 1$, as these cases corresponds to node-on-node contact.

As in the case of linear elements, it is possible to determine a pressure interpolation and associated integration scheme which are equivalent to the node-on-surface method T2N2.

**Proposition D.4** The formulations T21C2 and T2N2 are equivalent when the contact boundary integrals in the former are evaluated using the piecewise-trapezoidal rule over integration segments with endpoints defined by adjacent nodes.

**Proof:** With reference to the Figure 12, define integration segments whose endpoints are adjacent nodes. For example, one such segment would be between nodes $4I - 3$ and $4I - 2$, while another would be between nodes $4I - 2$ and $4I - 1$. In each segment, the boundary integrals are evaluated using the piecewise-trapezoidal rule. As in the proof of Proposition D.2, note that at any endpoint $K$ of a segment, only the pressure interpolation function $\Psi^a_K$ is non-zero. Letting $w_K$ be the integration weight at $K$, write

$$
\bar{B}^a_{KI} = -w_K \Phi^a_K(s^a_K)N^a(s^a_K).
$$

(D.17)

On the other hand, the node-on-surface formulation yields

$$
\hat{B}^a_{KI} = -\Phi^a_I(s^a_K)N^a(s^a_K),
$$

(D.18)
hence $C_{KL} = w_K \delta_{KL}$, where $[\delta_{KL}]$ is the $M \times M$ identity matrix. ■

One can easily deduce the following:

**Corollary D.2** The two-pass linear pressure interpolation for two-body contact using quadratic domain elements (method T21C2) fails the LBB condition when the contact boundary integrals are evaluated using the piecewise-trapezoidal rule over integration segments with endpoints defined by adjacent nodes.

**Proof:** This is an immediate consequence of Proposition D.4 and Proposition D.3. ■

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**APPENDIX E: Some results on the Babuška-Brezzi condition for two-body contact**

This appendix contains two propositions regarding the stability of specific finite element methods for the two-body contact problem.

**Proposition E.1** Method T11C2 fails the LBB condition for the reference analytical configuration.

**Proof:** Consider the reference analytical configuration of Figure 4, with piecewise linear interpolation in the domain and linear pressure interpolation on the contact boundary, as in Figure 7. Recall that

$$
\bar{q}_h^\alpha = \sum_{J=1}^{M} \bar{q}_J \Psi_J^\alpha(s^\alpha)
$$

and choose the pressure degrees-of-freedom $\bar{q}_I$ as alternating with absolute value $Q = 1$, that is

$$
\bar{q}_I = -\bar{q}_{I+1} = Q , \quad I = 1, \ldots, M - 1 .
$$

(E.1)

The displacement interpolation function $\Phi_I^\alpha$ associated with a typical node $I$ in the interior of $\Gamma_p^\alpha$ is non-zero over two contiguous element edges. Upon parametrizing $\Phi_I^\alpha$ over these two elements using a coordinate $\xi$, write

$$
\Phi_I^\alpha(\xi) = \begin{cases} 
1 + \xi & \text{for } \xi \in (-1, 0] \\
1 - \xi & \text{for } \xi \in (0, 1)
\end{cases} . \quad (E.2)
$$
Using the same coordinate, one may express the pressure interpolation as

\[
\tilde{q}_h(\xi) = \begin{cases} 
Q \frac{(1 + \beta) + 2\xi}{1 - \beta} & \text{for } \xi \in (-1, -\beta] \\
-Q \frac{\beta + 2\xi}{\beta} & \text{for } \xi \in (-\beta, 0] \\
Q \frac{(1 - \beta) + 2\xi}{1 - \beta} & \text{for } \xi \in (0, 1 - \beta] \\
Q \frac{(2 - \beta) - 2\xi}{\beta} & \text{for } \xi \in (1 - \beta, 1) 
\end{cases} \quad (E.3)
\]

It follows from the above that

\[
B^\alpha(\tilde{q}_h, \Phi^\alpha_\Gamma N^\alpha) = -Qh \left[ \int_{-1}^{-\beta} (1 + \xi) \frac{(1 + \beta) + 2\xi}{1 - \beta} d\xi + \int_{-\beta}^{0} (1 + \xi) \frac{\beta - 2\xi}{\beta} d\xi \\
+ \int_{0}^{(1-\beta)} (1 - \xi) \frac{(1 - \beta) + 2\xi}{1 - \beta} d\xi + \int_{(1-\beta)}^{1} (1 - \xi) \frac{(2 - \beta) - 2\xi}{\beta} d\xi \right]. \quad (E.4)
\]

Upon evaluating the integrals in (E.4), it is found that

\[
B^\alpha(\tilde{q}_h, \Phi^\alpha_\Gamma N^\alpha) = 0. \quad (E.5)
\]

The orthogonality condition (E.5) applies to all interpolation functions \( \Phi^\alpha_\Gamma \) sufficiently removed from the boundary of \( \Gamma_p \). Hence, by the arguments of Proposition 3.1, method T11C2 fails the LBB condition. ■

**Proposition E.2** Method T21C2 fails the LBB condition for the reference analytical configuration.

**Proof**: Consider again the reference analytical configuration of Figure 4 with pressure interpolations as in Figure 13. Let \( \tilde{q}_h \) be parametrized via

\[
\tilde{q}_h = \sum_{I=1}^{M} \tilde{q}_I \Psi_I(s),
\]

with the surface coordinate system chosen as \( s^\alpha = s^\beta = s \) and therefore \( \Psi^\alpha_I(s^\alpha) = \Psi^\beta_I(s^\beta) = \Psi_I(s) \). Consider the following choice for \( \tilde{q}_h \):

\[
\tilde{q}_{(4I-1)} = -\tilde{q}_{(4I-2)} = Q_1 \quad , \quad \tilde{q}_{(4I-3)} = -\tilde{q}_{(4I)} = Q_2 = \frac{1 + 3\beta}{1 + \beta} Q_1. \quad (E.6)
\]

As in Proposition E.1, for a typical shape function \( \Phi^\alpha_\Gamma \) in the interior of \( \Gamma_p^\alpha \), one needs to prove for this choice \( \tilde{q}_h \) that

\[
B^\alpha(\tilde{q}_h, \Phi^\alpha_\Gamma N^\alpha) = 0. \quad (E.7)
\]
First, consider equation E.7 for a typical corner node of a domain element in the interior of \( \Gamma_p \), say, node \( 4I - 2 \) in Figure (13). The shape function corresponding to this node attains non-zero value over the edges of elements \( \Omega_{2I-2} \) and \( \Omega_{2I} \) on the surface where node \( 4I - 2 \) lies, as well as over a region which includes parts of the edges of elements \( \Omega_{2I-3}, \Omega_{2I-1}, \) and \( \Omega_{2I+1} \) on the opposite surface. Now, proceed to parametrize this part of the contact region by \( \xi \), so that the width of a single element edge is \( 2h \), where \( h \) is the distance between any 2 successive nodes on a common surface. It follows that \( \tilde{q}_h \) can be defined for \( \xi \in (-2, 2) \) as

\[
\tilde{q}_h(\xi) = \begin{cases} 
-Q_1(1 - \frac{\xi + 2}{1 - \beta}) + Q_1 \frac{\xi + 2}{1 - \beta} & \text{for } \xi \in (-2, -1 - \beta] \\
Q_1(1 - \frac{\xi + 1 + \beta}{1 - \beta}) - Q_2 \frac{\xi + 1 + \beta}{\beta} & \text{for } \xi \in (-1 - \beta, -1] \\
-Q_2(1 - \frac{\xi + 1}{1 - \beta}) + Q_2 \frac{\xi + 1}{1 - \beta} & \text{for } \xi \in (-1, -\beta] \\
Q_2(1 - \frac{\xi + \beta}{\beta}) - Q_1 \frac{\xi + \beta}{\beta} & \text{for } \xi \in (-\beta, 0] \\
-Q_1(1 - \frac{\xi}{1 - \beta}) + Q_1 \frac{\xi}{1 - \beta} & \text{for } \xi \in (0, 1 - \beta] \\
Q_1(1 - \frac{\xi - 1 + \beta}{\beta}) - Q_2 \frac{\xi - 1 + \beta}{\beta} & \text{for } \xi \in (1 - \beta, 1] \\
-Q_2(1 - \frac{\xi - 1}{1 - \beta}) + Q_2 \frac{\xi - 1}{1 - \beta} & \text{for } \xi \in (1, 2 - \beta] \\
Q_2(1 - \frac{\xi - 2 + \beta}{\beta}) - Q_1 \frac{\xi - 2 + \beta}{\beta} & \text{for } \xi \in (2 - \beta, 2) . 
\end{cases}
\]  

(E.8)

Within the same interval, \( \Phi^p_{4I-2} \) can be written as

\[
\Phi_{4I-2}(\xi) = \begin{cases} 
\frac{1}{2} (\xi + 1)(\xi + 2) & \text{for } \xi \in (-2, 0] \\
\frac{1}{2} (\xi - 1)(\xi - 2) & \text{for } \xi \in (0, 2) ,
\end{cases}
\]  

(E.9)
so that

\[
B^\alpha(\bar{q}_h, \Phi^\alpha_{M-2} \mathbf{N}^\alpha) = -\frac{h}{2} \left[ \int_{-\beta}^{-1} \left( -Q_1(1 - \frac{\xi + 2}{1 - \beta}) + Q_1(\xi + 1)(\xi + 2) \right) d\xi 
+ \int_{-\beta}^{0} \left( -Q_2(1 - \frac{\xi + 1}{1 - \beta}) + Q_2(\xi + 1)(\xi + 2) \right) d\xi 
+ \int_{0}^{1} \left( Q_2(1 - \frac{\xi + 1}{1 - \beta}) - Q_2(\xi + 1)(\xi + 2) \right) d\xi 
+ \int_{1}^{1-\beta} \left( -Q_1(1 - \frac{\xi}{1 - \beta}) + Q_1(\xi - 1)(\xi - 2) \right) d\xi 
+ \int_{1-\beta}^{1-\beta} \left( -Q_2(1 - \frac{\xi}{1 - \beta}) + Q_2(\xi - 1)(\xi - 2) \right) d\xi 
+ \int_{1-\beta}^{1} \left( Q_2(1 - \frac{\xi - 2 + \beta}{\beta}) - Q_1(\xi - 2 + \beta)(\xi - 1)(\xi - 2) \right) d\xi \right].
\]

(E.10)

Evaluating the integral and taking into account (E.6) leads to

\[
B(\bar{q}_h, \Phi^\alpha_{M-2} \mathbf{N}^\alpha) = -\frac{h(1 - \beta)}{3} \left[ -Q_1(1 + 3\beta) + Q_2(1 + \beta) \right] = 0.
\]

(E.11)

A analogous calculation can be carried out for a typical center-edge node of a domain element in the interior of \(\Gamma_p\), for instance node 4I in Figure 13. The corresponding shape function \(\Phi^\alpha_{4I}\) is non-zero over the edge of element \(\Omega_{4I}\) where node 4I lies, as well as over a region which includes parts of the edges of elements \(\Omega_{(2I-1)}\) and \(\Omega_{(2I+1)}\) on the opposite surface. Again, using the same coordinate parametrization as in the preceding computation, it follows that \(\bar{q}_h\) can be written over the region \((-1, 1)\) as

\[
\bar{q}_h(\xi) = \begin{cases} 
-Q_1(1 - \frac{\xi + 1}{1 - \beta}) + Q_1(\xi + 1) & \text{for } \xi \in (-1, -\beta] \\
Q_1(1 - \frac{\xi}{1 - \beta}) - Q_2(\xi + 1) & \text{for } \xi \in (-\beta, 0] \\
-Q_2(1 - \frac{\xi}{1 - \beta}) + Q_2(\xi) & \text{for } \xi \in (0, 1 - \beta] \\
Q_2(1 - \frac{\xi - 2 + \beta}{\beta}) - Q_1(\xi - 2 + \beta) & \text{for } \xi \in (1 - \beta, 1) .
\end{cases}
\]

(E.12)

In addition, \(\Phi^\alpha_{4I}\) is given by

\[
\Phi_{4I} = 1 - \xi^2 ,
\]
over the same region. The integral $B^\alpha(q_h, \Phi^\alpha_{4I} N^\alpha_r)$ now becomes

$$B^\alpha(q_h, \Phi^\alpha_{4I} N^\alpha_r) = -h \left[ \int_{-1}^{-\beta} (-Q_1(1 - \xi + \frac{1}{1 - \beta}) + Q_1 \frac{1}{1 - \beta} (1 - \xi^2) d\xi \\
\int_{-\beta}^{0} (Q_1(1 - \frac{\xi + \beta}{\beta}) - Q_2 \frac{\xi + \beta}{\beta} (1 - \xi^2) d\xi \\
+ \int_{0}^{1 - \beta} (-Q_2(1 - \frac{\xi}{1 - \beta}) + Q_2 \frac{\xi}{1 - \beta} (1 - \xi^2) d\xi \\
+ \int_{1 - \beta}^{1} (Q_2(1 - \frac{\xi - 1 + \beta}{\beta}) - Q_1 \frac{\xi - 1 + \beta}{\beta} (1 - \xi^2) d\xi \right] , \quad (E.13)$$

and, upon evaluation and use of (E.6), leads to

$$B^\alpha(q_h, \Phi^\alpha_{4I} N^\alpha_r) = -h \left[ \frac{(1 - \beta)}{6} \left[ Q_1(1 + 3\beta) - Q_2(1 + \beta) \right] \right] = 0 . \quad (E.14)$$

As in Proposition E.1, there exist nodes $I$ in the neighborhood of the edge of contact where $B^\alpha(q_h, \Phi^\alpha_{4I} N^\alpha_r) \neq 0$. However, since the number of these nodes is finite and bounded independently of $h$, Proposition 3.1 implies that method T21C2 fails the LBB condition. ■

**APPENDIX F: Babuška-Brezzi condition for an alternative linear pressure interpolation**

This appendix contains the proofs of satisfaction of the LBB condition for the two-body problem using an alternative two-pass piecewise linear pressure interpolation in conjunction with piecewise linear and quadratic domain elements.

**Proposition F.1** The modified two-pass formulation based on piecewise linear domain elements and the continuous piecewise linear pressure interpolation shown in Figure 14 satisfies the LBB condition for the reference analytical configuration.

**Proof:** Given a pressure field $q_h$, defined by the degrees-of-freedom $q_{2I}$, $q_{2I+1}$ and $q_{2I+2}$ along the surface, choose

$$\tilde{v}_{4I-2} = \tilde{v}_{4I+1} = \tilde{v}_{4I+2} = 0 ,$$

$$\tilde{v}_{4I-1} = q_{2I} N^\alpha , \quad \tilde{v}_{4I} = -q_{2I+1} N^\alpha , \quad \tilde{v}_{4I+3} = q_{2I+2} N^\alpha ,$$

(F.1)
where $\mathbf{N}^\alpha$ refers to the outward unit normal to the surface defined by nodes $4I - 2$, $4I$, and $4I + 2$. Given (F.1), it is clear that there exists a constant $k_1 > 0$, such that

$$\|\overline{v}_h\|_{(\alpha, \Gamma_p)} \leq k_1 \|q_h\|_{(0, \Gamma_p)} .$$

With reference to Figure 14, consider the boundary region corresponding to the edges of elements $\Omega_{4I-2}$ and $\Omega_{4I}$, and introduce a surface coordinate $\xi \in (0, 2)$ that spans this region. It follows that

$$B(q_h, \overline{v}_h) = h \left[ \frac{1}{2 - \beta} \int_0^{1-\beta} (q_{2I}(2 - \beta - \xi) + q_{2I+1}(\xi))(q_{2I}(\xi + \beta) + q_{2I+1}(\xi)) \, d\xi ight. \\
+ \frac{1}{(2 - \beta)} \int_{1-\beta}^{1} (q_{2I}(2 - \beta - \xi) + q_{2I+1}(\xi))(q_{2I}(2 - \beta - \xi) + q_{2I+1}(\xi)) \, d\xi \\
+ \frac{1}{(2 - \beta)} \int_{1}^{2-\beta} (q_{2I}(2 - \beta - \xi) + q_{2I+1}(\xi))(q_{2I}(2 - \beta - \xi) + q_{2I+1}(2 - \xi)) \, d\xi \\
+ \frac{1}{\beta} \int_{2-\beta}^{2} (q_{2I+2}(2 - \xi) + q_{2I+2}(\xi - 2 + \beta))(q_{2I+1}(2 - \xi) + q_{2I+2}(\xi - 2 + \beta)) \, d\xi \right] .$$

Upon performing the integration, one finds that

$$B(q_h, \overline{v}_h) = \frac{h}{6(2 - \beta)} \left[ (q_{2I}^2 + q_{2I+1}^2)(6\beta - 6\beta^2 + \beta^3) + (q_{2I} + q_{2I+1})^2(6 - 6\beta + \beta^3) \right] \\
+ \frac{h\beta^2}{6} \left[ (q_{2I+1} + q_{2I+2})^2 + q_{2I+1}^2 + q_{2I+2}^2 \right] .$$

Over the same interval, one can easily show that

$$\int_0^{2h} q_h^2 \, dx = \frac{(2 - \beta)h}{6} \left[ (q_{2I} + q_{2I+1})^2 + (q_{2I} + q_{2I+1})^2 \right] \\
+ \frac{h\beta}{6} \left[ (q_{2I+1} + q_{2I+2})^2 + (q_{2I+1} + q_{2I+2})^2 \right] .$$

Hence, it follows that

$$B(q_h, \overline{v}_h) \geq \beta \|q_h\|_{(0, \Gamma_p)}^2 .$$

Therefore, appealing to Proposition 3.2 the LBB condition is satisfied for the reference analytical configuration.

The preceding analysis can be repeated for the special case $\beta = 0$ corresponding to node-on-node contact. In this case, the theorem can be shown to hold true by requiring the pressure variables at coincident nodes to attain the same value.
**Proposition F.2** The modified two-pass formulation based on piecewise quadratic domain elements and the continuous piecewise linear pressure interpolation shown in Figure 15 satisfies the LBB condition for the reference analytical configuration.

**Proof:** With reference to Figure 15, given any admissible \( q_h \), choose \( \tilde{v}_h \) such that

\[
\tilde{v}_{4I-3} = \tilde{v}_{4I-2} = \tilde{v}_{4I+1} = 0
\]

\[
\tilde{v}_{4I-1} = q_{2I} N^a , \tilde{v}_{4I} = -q_{2I+1} N^a ,
\]

where \( N^a \) refers to the outward unit normal to the surface defined by nodes \( 4I - 2, 4I, \) and \( 4I + 2 \). It is easy to discern that there exists a constant \( k_2 > 0 \), such that

\[
\| \tilde{v}_h \|_{(0, \Gamma_p)} \leq k_2 \| q_h \|_{(0, \Gamma_p)} .
\]

Satisfaction of the LBB condition is therefore dependent on the evaluation of the integral \( B(q_h, \tilde{v}_h) \). Taking the unit cell to be from \( x = 0 \) to \( x = 2h \) and using, again, the surface coordinate \( \xi \in (0, 2) \), this integral can be written as

\[
B(q_h, \tilde{v}_h) = \frac{h}{2-\beta} \int_0^{2-\beta} \left( q_{2I} (2-\beta-\xi) + q_{2I+1}(\xi) \right) \\
\left( q_{2I} (1-(\xi-1+\beta)^2) + q_{2I+1} (1-(\xi-1)^2) \right) d\xi \\
+ \frac{h}{\beta} \int_0^{\beta} \left( q_{2I+1}(\beta-\xi) + q_{2I+2}(\xi) \right) \\
\left( q_{2I+1} (1-(\xi+(1-\beta)^2) + q_{2I+2} (1-(\xi-1)^2) \right) d\xi .
\]

Evaluating the integral leads to

\[
B(q_h, \tilde{v}_h) = \frac{h(2-\beta)^2}{12} \left[ (2+\beta)(q_{2I} + q_{2I+1})^2 + 2\beta(q_{2I}^2 + q_{2I+1}^2) \right] \\
+ \frac{h\beta^2}{12} \left[ (4-\beta)(q_{2I+1} + q_{2I+2})^2 + (4-2\beta)(q_{2I+1}^2 + q_{2I+2}^2) \right] .
\]

Over the same interval, it is simple to show that

\[
\int_0^{2h} q_h \, dx = \frac{h(2-\beta)}{6} \left[ (q_{2I} + q_{2I+1})^2 + (q_{2I}^2 + q_{2I+1}^2) \right] \\
+ \frac{\beta h}{6} \left[ (q_{2I+1} + q_{2I+2})^2 + (q_{2I+1}^2 + q_{2I+2}^2) \right] .
\]

It follows that

\[
B(q_h, \tilde{v}_h) \geq (2-\beta)\beta \| q_h \|_{(0, \Gamma_p)}^2 .
\]

Proposition 3.2, in conjunction with equations (F.8) and (F.12) yields the proof. ■

As in Proposition F.1, the limiting cases \( \beta = 0 \) and \( \beta = 2 \) do not alter the outcome of the preceding analysis.
Figure 1: A canonical two-dimensional Signorini problem.
Figure 2: Signorini problem with linear domain elements and formulations (S10D1), (S11D1), (S10D2), and (S11C1).
Figure 3: Signorini problem with quadratic domain elements and formulations (S20D1), (S21D1), (S21C1), and (S22C1).
Figure 4: Reference analytical configuration for two-body contact.
Figure 5: Two-body contact problem with linear domain elements and one-pass continuous linear pressure over each element edge (T11C1).
Figure 6: Two-body contact problem with linear domain elements and one-pass node-on-surface resultants (T1N1).
Figure 7: Two-body contact problem with linear domain elements and two-pass continuous linear pressure over each contact segment (T11C2).
Figure 8: Two-body contact problem with linear domain elements and two-pass node-on-surface resultants (T1N2).
Figure 9: Two-body contact problem with quadratic domain elements and one-pass continuous linear pressure over each element edge (T21C1).
Figure 10: Two-body contact problem with quadratic domain elements and one-pass continuous quadratic pressure over each element edge (T22C1).
Figure 11: Two-body contact problem with quadratic domain elements and one-pass node-on-surface resultants (T2N1).
Figure 12: Two-body contact problem with quadratic domain elements and two-pass node-on-surface resultants (T2N2).
Figure 13: Two-body contact problem with quadratic domain elements and two-pass continuous linear pressure over each contact segment (T21C2).
Figure 14: Two-body contact problem with linear domain elements and modified two-pass continuous linear pressure over each contact segment (T11MC2).
Figure 15: Two-body contact problem with quadratic domain elements and modified two-pass continuous linear pressure over each contact segment (T21MC2).
Figure 16: Two-layered thick-walled cylinder under external pressure: reference configuration, loading, and boundary conditions.
Figure 17: Two-layered thick-walled cylinder under external pressure: convergence of the contact pressure in the $L_2$-norm under $h$-refinement (T11MC2).
Figure 18: Two-layered thick-walled cylinder under external pressure: convergence of the contact pressure in the $L_2$-norm under $h$-refinement (T11MC2).