On the use of consistent approximations in boundary element-based shape optimization in the presence of uncertainty

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Abstract

This work addresses certain aspects of shape optimization for linearly elastic systems in the presence of uncertainty. The goal is to formulate and test a set of computationally tractable methods for incorporating uncertainty in the system. The boundary element method is employed in solving the underlying elasticity equations. The resulting minimax problem is studied in the context of the so-called consistent approximations, which allow for adaptive control of the discretization error during the iterative optimization process.

Keywords: Boundary element method, shape optimization, minimax problem, consistent approximations.

1 Introduction

The presence of uncertainties has important implications in the response of many engineering systems. As a result, engineering design is increasingly expected to account for such uncertainties, see [1] and [2] for a typical example from the aerospace industry. The simplest way to incorporate uncertainty in engineering design is to consider the expected value of a given cost function and its variations over a range of parametric values defined in a stochastic model, see, e.g., [3–5]. While such methods are useful, they generally tend to be inefficient when only the bounds on the uncertainty parameter are known or when the problem is to determine the design that minimizes the worst-case outcome with respect to a given cost function.

This article concerns a minimax formulation for the robust design of systems whose response is governed by the equations of linear elastostatics, and for which there exist bounds for all uncertain data. Particular emphasis is placed on the problem of shape optimization. The boundary element method is employed to solve the equations of elastostatics, owing to

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its versatility in discretizing the evolving boundary of the domain. The principal novelty in this work is the application of an efficient automatic adaptive precision optimization algorithm based on the concept of consistent approximations to reduce the computational cost and control the errors during the optimization process. Using consistent approximations, as originally introduced in [6], the optimization problem is solved iteratively using initially very coarse discretizations. These are successively refined (or, in certain cases, derefined) to facilitate convergence of the optimization algorithm. In this way, the overall cost of solving the optimization problem is reduced, as the bulk of the computations is performed on relatively coarse discretizations. Adaptive discretizations based on the concept of consistent approximations have been used in connection with finite element solutions of boundary-value problems [7], as well as in problems of optimal control [8].

The organization of the article is as follows: The abstract optimization problem and the governing field equations are reviewed in Section 2. Certain essential aspects of the boundary element method including relevant convergence results are included in Section 3. These are followed by the formulation of the minimax problem in Section 4 and the application of the consistent approximations concept in the context of boundary elements in Section 5. Finally, certain numerical aspects of the implementation and representative examples are included in Section 6 in the context of robust shape optimization.

2 Problem statement and notation

2.1 Description of the system

Consider the shape optimization of mechanical and structural systems made of homogeneous, isotropic linearly elastic material in the presence of uncertainty. The design parameters are the shape of part of the boundary of the system. Uncertainty is introduced in the model using geometric parameters for which only a range of values is known. The objective here is to obtain a robust design, namely one that minimizes the worst-case outcome. This design corresponds to the configuration that minimizes over the set of design variables the maximum value of the cost function obtainable for the given range of variation of the uncertainty parameters. Mathematically, the robust design is associated with the solution of a minimax problem of the form

$$\min_{x \in X} \max_{y \in Y} f(x, y) , \quad (1)$$
where \( f(\cdot, \cdot) \) is the cost function, \( x \) is the design vector and \( y \) is the uncertainty parameter vector, defined on sets \( X \subset \mathbb{R}^n \) and \( Y \subset \mathbb{R}^m \), respectively.

The canonical problem used to illustrate the proposed method aims at minimizing the deformation of a two-dimensional plate with circular holes, when subjected to mixed Dirichlet/Neumann boundary conditions. Here, the design variables are the positions of the holes, while the uncertainty stems from the potentially inexact placement of some or all of the holes, see Figure 1(a). The desired measure of deformation is defined as the average displacement of a portion \( \Gamma_r \) of the boundary, in the form

\[
f(z) = \int_{\Gamma_r} |v(z; s)|^2 ds ,
\]

where \( z \) is a vector comprised of the design and uncertainty variables, i.e., \( z \in Z = X \times Y \subset \mathbb{R}^{n+m} \). Also, \( v(z; s) \) denotes the displacement of a point as a function of a boundary coordinate \( s \), for given value of \( z \).

In the remainder of this article, the following notational conventions are employed: a bold character, e.g., \( \mathbf{a} \) will denote a vector quantity with components \( a^j \), \( a_N \) will denote an element of a scalar-valued sequence, and \( \mathbf{a}_N \) an element of a vector-valued sequence.

### 2.2 Governing equations

In the absence of body forces, the equations of equilibrium for a homogeneous and isotropic linearly elastic solid are given by

\[
-\Delta^* v = 0 \quad \text{in} \quad \Omega ,
\]

where \( \Omega \) is the region occupied by the body in its reference configuration. Also, the Lamé operator \( \Delta^* \) is defined as

\[
-\Delta^* v \defeq \mu \Delta v + (\lambda + \mu) \text{grad div } v ,
\]

where \( \lambda \) and \( \mu \) are elastic constants, and \( \Delta \) is the Laplacian operator. The traction vector \( \mathbf{t} \) on a surface with outward unit normal \( \mathbf{n} \) can be expressed as \( \mathbf{t} = T v \), where the traction operator \( T \) is defined by

\[
T v = \lambda (\text{div } v) \mathbf{n} + 2\mu v_{,n} + \mu \mathbf{n} \times \text{curl } v
\]

and \( v_{,n} \) denotes the derivative of \( v \) along \( \mathbf{n} \). The boundary conditions are

\[
\mathbf{t} = \mathbf{t}_0 \quad \text{on} \quad \Gamma_t ,
\]

\[
v = \mathbf{v}_0 \quad \text{on} \quad \Gamma_v ,
\]
where \( \bar{t} \) and \( \bar{v} \) are the applied tractions and displacements on \( \Gamma_t \) and \( \Gamma_v \), while the boundary \( \partial \Omega \) of \( \Omega \) satisfies \( \partial \Omega = \Gamma_t \cup \Gamma_v \).

It is noted that, where possible without loss of clarity, the dependency of \( \Omega \) and \( \partial \Omega \) on \( z \) is not explicitly declared.

3 Boundary element model

In the context of shape optimization, the use of the boundary element method is particularly attractive due to the convenience in treating the evolution of the geometry without the cost of domain remeshing and the existence of analytical and semi-analytical expressions for the sensitivity, [9,10]. The boundary element method can make the computations associated with the solution of the minimax problem (1) inexpensive as shown in Section 6.

3.1 From the boundary-value problem to the algebraic system

Using the direct approach to the boundary element method, boundary integral equations are derived based on the Betti-Somigliana representation formulation, see, e.g., [11]. For the differential equation (3), this gives rise to

\[
\mathbf{v}(\zeta) = \mathbf{V} \mathbf{\sigma}(\zeta) - \mathbf{W} \phi(\zeta), \quad \zeta \in \Omega, \tag{7}
\]

where \( \mathbf{\sigma}(\zeta) = T \mathbf{v}(\zeta)|_{\partial \Omega} \) and \( \phi(\zeta) = \mathbf{v}(\zeta)|_{\partial \Omega} \). Also, \( \mathbf{V} \) and \( \mathbf{W} \) are the simple and double-layer potentials defined by

\[
\mathbf{V} \mathbf{\sigma}(\zeta) = \int_{\partial \Omega} \mathbf{E}(\zeta, \xi) T \mathbf{v}(\xi) d\xi \quad \tag{8}
\]

\[
\mathbf{W} \phi(\zeta) = \int_{\partial \Omega} \left( \frac{dT}{d\xi} \mathbf{E}(\zeta, \xi) \right)^T \mathbf{v}(\xi) d\xi, \tag{9}
\]

where \( \mathbf{E}(\zeta, \xi) \) is the fundamental solution of (3). Upon applying the trace operator to (7), one finds that

\[
\mathbf{V} \mathbf{\sigma}(\gamma) = \left( \frac{1}{2} \mathbf{I} + \mathbf{K} \right) \phi(\gamma), \tag{10}
\]

where \( \gamma \in \partial \Omega \), and \( \mathbf{V} \) and \( \mathbf{K} \) are two boundary integral operators defined by

\[
\mathbf{V} \mathbf{\sigma}(\gamma) \triangleq \lim_{\Omega \ni \zeta \rightarrow \gamma \in \partial \Omega} \mathbf{V} \mathbf{\sigma}(\zeta), \quad \mathbf{K} \phi(\gamma) \triangleq \lim_{\Omega \ni \zeta \rightarrow \gamma \in \partial \Omega} \mathbf{W} \phi(\zeta) + \frac{1}{2} \phi(\zeta), \tag{11}
\]

and \( \mathbf{I} \) is the identity operator. For this interior problem, equation (10) can be expressed in the general form

\[
\mathcal{A} \mathbf{u} = \mathbf{b} \text{ on } \partial \Omega, \quad \tag{12}
\]
where \( u \) is the unknown displacement or traction on the boundary \( \partial \Omega \). For Dirichlet boundary conditions

\[
Au = V \sigma , \quad b = \left( \frac{1}{2} I + K \right) \phi , \tag{13}
\]

which is a Fredholm integral equation of the first kind, while for Neumann boundary conditions

\[
Au = \left( \frac{1}{2} I + K \right) \phi , \quad b = V \sigma , \tag{14}
\]

which is a Cauchy-singular integral equation.

Equation (12) is discretized using boundary elements. In this case, upon using standard numerical integration rules, the discrete counterpart of (12) reduces to an algebraic system of the form

\[
F(z)v = G(z)t , \tag{15}
\]

where \( v, t \) are the vectors of nodal displacements and tractions and \( F(\cdot), G(\cdot) \) are influence matrices, see [12], [13] for further details. As noted in (15), the matrices \( F \) and \( G \) are functions of a vector \( z \). The components \([F_{ij}]\) and \([G_{ij}]\) are respectively associated with the traction and displacement of the \( j \)-th degree of freedom due to a unit load at the \( i \)-th degree of freedom. Upon applying the boundary conditions (6), equation (15) gives rise to

\[
A(z)u_N = b(z) , \tag{16}
\]

where \( u_N \) is the vector of nodal unknowns quantities, displacements and tractions on the boundary \( \partial \Omega \) and \( b \) is the loading vector.

### 3.2 Convergence of the boundary element method

In this section, some important results concerning convergence of the boundary element method are reviewed. Using a collocation-boundary element formulation with an interpolating spline of order \( d \), the following theorem, [14, Theorem 10.3.2], gives an error estimate for the boundary-value problem (3-6):

**Theorem 3.1** Let \( A \) be a strongly elliptic pseudodifferential operator of order \( \beta \), and assume that \( A : H^r(\partial \Omega) \to H^{r-\beta}(\partial \Omega) \) is an isomorphism for all \( r \in \mathbb{R} \). Let \( u \in H^t \) and \( u_N \in S_N^d \) (\( \subset H^t \)) satisfy the collocation equations

\[
Au = Au_N \text{ on } \partial \Omega_N , \tag{17}
\]
where $\partial \Omega_N$ is the boundary element mesh and $N$ is a discretization parameter that satisfies $N \sim 1/h$, $h$ being a characteristic mesh size. Then, there exists a $C > 0$ such that
\[
\|u - u_N\|_{H^s} \leq C \left( \frac{1}{N} \right)^{1-s} \|u\|_{H^1},
\] (18)
for all $\beta$, $d$, $s$, $t$, $\in \mathbb{R}$, such that $\beta < d$ if $d$ is odd, $\beta < d + 1$ if $d$ is even, and $\beta \leq s \leq t \leq d + 1$, $s < d + 1/2$, $\beta + 1/2 < t$.

In the present problem, let $u \in H^2(\Omega)$, hence $u \in H^{1.5}(\partial \Omega)$ and $t = 1.5$. Then, using second-order interpolating splines, i.e., $d = 2$, one finds that $\beta \leq s \leq 1.5$ and $\beta < 1$, which, upon invoking Theorem 3.1 leads to
\[
\|u - u_N\|_{H^s} \leq C \left( \frac{1}{N} \right)^{1.5-s} \|u\|_{H^{1.5}}.
\] (19)
In the above, the fractional norms $H^s(\partial \Omega)$ are defined in one of the several equivalent ways, see, e.g., [15].

The value of $\beta$ depends on the nature of the problem, see [11]. Interior displacement problems can either lead to Fredholm integral equations of the first kind ($\beta < 0$) or Cauchy-singular integral equations ($\beta = 0$), while interior traction problems lead either to hypersingular integral equations ($\beta > 0$) or Cauchy-singular integral equations depending on the formulation. For the Fredholm integral equation (13) and Cauchy-singular integral equation (14), where $\beta \leq 0$, equation (19) readily leads to
\[
\|u - u_N\|_2 \leq C \left( \frac{1}{N} \right)^{1.5},
\] (20)
where $C$ is a positive constant.

3.2.1 The cost function

By inspection, the cost $f(z) \triangleq f(x, y)$ is a composite function of the form $f(z) = \tilde{f}(u(z))$, where $\tilde{f}(u(z)) = \int_{\Gamma_r} [u(z; s)]^2 ds$. Taking $\Gamma_r$ to be a Neumann boundary, the function $u(z; s)$ denotes the displacement of a point with coordinate $s$ for a given $z$ in the (assumed single) direction of the applied traction. Next, define a numerical approximation to $f(\cdot)$ by evaluating (2) for the numerical solution by the trapezoidal rule, such that, for any $u_L \in S^L_d$, with $L$ and $N$ positive integers,
\[
\tilde{f}_N(u_L(z)) \triangleq \sum_{i=1}^{N} \frac{h}{2} \left( [u_L(z; s_i)]^2 + [u_L(z; s_{i+1})]^2 \right),
\] (21)
where \( l(\Gamma_r) \) the length of \( \Gamma_r \), and \( h = l(\Gamma_r)/N \) is the size of a boundary element on \( \Gamma_r \). In order to simplify the error analysis, the trapezoidal rule in (21) is applied separately to each boundary element domain. This implies that all exterior boundary element nodes are also sampling points of the trapezoidal rule.

From a standard result in numerical analysis, since \( u_L \) is at least of class \( C^2 \) on each element,
\[
|\tilde{f}_N(u_L(z)) - \hat{f}(u_L(z))| \leq M \frac{1}{N^2},
\]
for some constant \( M > 0 \), which implies that
\[
\lim_{N \to \infty} \tilde{f}_N(u_L(z)) = \hat{f}(u_L(z)).
\]
Also, from equation (20),
\[
\lim_{L \to \infty} ||u_L(z) - u(z)||_2 = 0.
\]
Therefore, since \( f(\cdot) \) is a continuous function, \( \tilde{f}(u_L(z)) \to \hat{f}(u(z)) \) as \( L \to \infty \), so that
\[
\lim_{N \to \infty} \lim_{L \to \infty} \tilde{f}_N(u_L(z)) = \lim_{N \to \infty} \tilde{f}_N(u(z)) = \hat{f}(u(z)).
\]
Then, letting \( f_N(z) \triangleq \tilde{f}_N(u_N(z)) \),
\[
\lim_{N \to \infty} f_N(z) = f(z), \quad \forall \ z \in Z.
\]

### 3.2.2 Sensitivity of the cost function

The cost function (2) and its discrete counterpart (21) are quadratic in the solution vector \( u \) on \( \Gamma_r \). In particular,
\[
\nabla f(u) = 2 \int_{\Gamma_r} [\nabla u(z; s)] [u(z; s)] \, ds,
\]
where \( \nabla u \triangleq \frac{du}{dz} \). For the purpose of exploring the sensitivity of the cost function, one may employ the trapezoidal rule to evaluate \( [\nabla u] \) \( u \). Assuming that \( u(z; \cdot) \) and \( \nabla u(z; \cdot) \) are at least of class \( C^2 \) for all \( z \in Z \), it follows that
\[
\lim_{N \to \infty} \nabla \tilde{f}_N(u(z)) = \nabla \tilde{f}(u(z)).
\]

**Theorem 3.2** Let \( u \) be differentiable in \( z \) and satisfy the condition \( \nabla u \in H^1(\partial\Omega) \). If the prescribed boundary loads are also sufficiently smooth on \( \partial\Omega \), then
\[
\lim_{N \to \infty} \nabla u_N(z) = \nabla u(z).
\]
Proof: Differentiating the boundary integral equation (10) with respect to \( z \) leads to

\[
V_{;z} \sigma + V_{;z} \phi = K_{;z} \phi + \left( \frac{1}{2} I + K \right) \phi_{;z} .
\]  

(30)

Since equation (10) has been already solved, one can write equation (30) in the general form

\[
\mathcal{A}u_{;z} = b_{;z} \text{ on } \partial \Omega .
\]  

(31)

Here,

\[
\mathcal{A}u_{;z} = V_{;z} \sigma , \text{ and } b_{;z} = K_{;z} \phi + \left( \frac{1}{2} I + K \right) \phi_{;z} - V_{;z} \sigma ,
\]  

(32)

when \( \phi_{;z} \) is prescribed on \( \partial \Omega \), and

\[
\mathcal{A}u_{;z} = \left( \frac{1}{2} I + K \right) \phi_{;z} , \text{ and } b_{;z} = V_{;z} \sigma + V_{;z} \sigma - K_{;z} \phi ,
\]  

(33)

when \( \sigma_{;z} \) is prescribed on \( \partial \Omega \).

This implies that the boundary integral operators for the sensitivity analysis are identical to those of the original system (12). Also, recalling that \( E_{;z} (\zeta, \xi) \) and \( TE_{;z} (\zeta, \xi) \) have the same order of singularity as \( E(\zeta, \xi) \) and \( TE(\zeta, \xi) \), see [10,16], one may conclude that the boundary term \( b_{;z} \) does not pose any problem if the prescribed boundary fields are sufficiently smooth. Therefore, Theorem 3.1 applies to the sensitivity problem, hence

\[
\left\| \frac{du}{dz_i} - \frac{du_N}{dz_i} \right\|_{H^s} \leq C \left( \frac{1}{N} \right)^{t-s} \left\| \frac{du}{dz_i} \right\|_{H^t} ,
\]  

(34)

for \( i = 1, \ldots, n + m \) and for all \( \beta, d, s, t, \in \mathbb{R} \), such that \( \beta < d \) if \( d \) is odd, \( \beta < d + 1 \) if \( d \) is even, and \( \beta \leq s \leq t \leq d + 1, \; s < d + 1/2, \; \beta + 1/2 < t \). Therefore, if \( \frac{du}{dz_i} \) is bounded in the \( H^t \) norm for \( i = 1, \ldots, n + m \), then equation (29) holds.

Finally, using equations (24), (28) and (29), coupled with the fact that \( \nabla f(\cdot) \) and \( \nabla f_N(\cdot) \) are continuous for all \( N \in \mathbb{N} \), implies that

\[
\lim_{N \to \infty} \nabla f_N(z) = \nabla f(z) , \; \forall \; z \in \mathbb{Z} .
\]  

(35)

4 Minimax problem

4.1 Abstract formulation of the design problem

To simplify the original minimax problem (1), assume that it can be converted into a problem of the form

\[
\min_{x \in X} \psi(x) = \min_{x \in X} \max_{j \in q} f_j(x) ,
\]  

(36)

where
\[ f^j(x) \triangleq f(x, y_j), \] (37)
and \( q = \{1, 2, \ldots, q\} \). The number of points \( y_j \) in \( Y \), as well as their coordinates, need to be chosen so that they accurately capture the variation of the function \( f \) over \( Y \). In the unconstrained case \( X = \mathbb{R}^n \), in the constrained case,
\[ X \triangleq \{ x \in \mathbb{R}^n \mid c^k(x) \leq 0, \ k = 1, \ldots, p \}, \] (38)
where the functions \( c^k : \mathbb{R}^n \to \mathbb{R} \) are continuously differentiable.

4.2 Optimality function for the unconstrained problem

Consider the problem
\[ P_u \min_{x \in \mathbb{R}^n} \max_{j \in q} f^j(x) . \] (39)
Suppose that \( \hat{x} \) is an optimal solution of (39). Then, see [6, Theorem 2.1.1.], a first-order optimality condition is given by
\[ d\psi(\hat{x}; h) \geq 0, \ \forall \ h \in \mathbb{R}^n, \] (40)
where \( d\psi(\cdot; h) \) is the directional derivative of \( \psi(\cdot) \) in the direction \( h \), such that
\[ d\psi(x; h) = \max_{j \in \hat{q}(x)} \langle \nabla f^j(x), h \rangle, \] (41)
where \( \hat{q}(x) \triangleq \{ j \in q \mid f^j(x) = \psi(x) \} \). Equivalently, the relation (40) holds at \( \hat{x} \) if, and only if, \( 0 \in \partial \psi(\hat{x}) \), where the subgradient \( \partial \psi(\hat{x}) \) is given by
\[ \partial \psi(\hat{x}) \triangleq \text{co} \left\{ \nabla f^j(\hat{x}) \right\}. \] (42)

Define the optimality function \( \theta_u(\cdot) \) as
\[ \theta_u(x) = \min_{h \in \mathbb{R}^n} \left\{ \tilde{\psi}(x, h) - \psi(x) \right\}, \] (43)
where \( \tilde{\psi}(x, h) \) is given by a first-order Taylor series expansion of \( f^j(x + h) \) around \( x \) with an additional quadratic term in \( ||h||_2^2 \), namely
\[ \tilde{\psi}(x, h) = \max_{j \in q} \{ f^j(x) + \langle \nabla f^j(x), h \rangle + \frac{1}{2} \delta ||h||_2^2 \}. \] (44)
Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$ and $\delta > 0$ is a parameter whose value lies between the maximum and minimum eigenvalue of all the $\nabla^2 f^j(\cdot)$. Substituting (44) into (43) one finds that $\theta_u(\cdot)$ can be expressed equivalently as

$$\theta_u(x) = -\min_{\mu \in \Sigma_q} \left\{ \sum_{j=1}^{q} \mu^j [\psi(x) - f^j(x)] + \frac{1}{2\delta} \left\| \sum_{j=1}^{q} \mu^j \nabla f^j(x) \right\|^2 \right\} , \tag{45}$$

where $\Sigma_q$ is the unit simplex in $\mathbb{R}^q$ given by

$$\Sigma_q = \left\{ \mu \in \mathbb{R}^q \mid \sum_{j=1}^{q} \mu^j = 1, \; \mu^j \geq 0, \; j = 1, \ldots, q \right\} . \tag{46}$$

And the search direction associated with $\theta(x)$ by

$$h(x) = -\frac{1}{\delta} \sum_{j=1}^{q} \hat{\mu}^j \nabla f^j(x) , \tag{47}$$

where $\hat{\mu}$ is a solution of (45). The search direction $h(\cdot)$ is continuous and satisfies the relation $\psi(x; h(x)) \leq \theta_u(x)$.

From equation (45), it is clear that $\theta_u(x) \leq 0$ and $\theta_u(\hat{x}) = 0$ if, and only if,

$$\sum_{j=1}^{q} \mu^j [\psi(\hat{x}) - f^j(\hat{x})] = 0 , \tag{48}$$

$$\sum_{j=1}^{q} \mu^j \nabla f^j(\hat{x}) = 0 , \tag{49}$$

which implies that $\theta_u(\hat{x}) = 0$ if, and only if, $0 \in \partial \psi(\hat{x})$.

### 4.3 Optimality function for the constrained problem

In the presence of inequality constraints, i.e., when $X = \{ x \mid c^k(x) \leq 0, \; k \in p \}$ where $p \triangleq \{ 1, 2, \ldots, p \}$, problem (36) assumes the form

$$\mathcal{P}_c \min_{x \in \mathbb{R}^n} \left\{ \psi(x) \mid c^k(x) \leq 0, \; k \in p \right\} . \tag{50}$$

A locally equivalent minimax problem near a solution $\hat{x}$ is given by (see [6, Definition 2.2.1])

$$\min_{x \in \mathbb{R}^n} \hat{F}(x) , \text{ with } \hat{F}(x) \triangleq \max\{ \psi(x) - \psi(\hat{x}) , c(x) \} , \tag{51}$$

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\(^1\)See [6, Theorem 2.1.6]
where \( c(x) = \max_{k \in p} c^k(x) \) is the constraint violation function. Since (51) is a minimax problem, the above optimality conditions apply to it. Because \( \tilde{F}(\cdot) \) requires the advance knowledge of \( \hat{x} \), it is approximated by the function \( F(x, x + h) \) given by

\[
F(x, x + h) \triangleq \max \left\{ \psi(x + h) - \psi(x) - \eta c(x)^+ , \ c(x + h) - c(x)^+ \right\},
\]

with \( \eta > 0^2 \). The function \( F(\cdot, \cdot) \) is approximated at \((x, x + h)\) by \( \tilde{F}(\cdot, \cdot) \), where

\[
\tilde{F}(x, x + h) \triangleq \max \left\{ \tilde{\psi}(x, h) - \psi(x) - \eta c(x)^+ , \ \tilde{c}(x, h) - c(x)^+ \right\} + \frac{1}{2} \delta ||h||^2,
\]

where \( \tilde{\psi}(\cdot, \cdot) \) is given by equation (44), \( c(x)^+ = \max \{c(x), 0\} \), and \( \tilde{c}(\cdot, \cdot) \) is a first-order approximation of \( c(\cdot) \). Hence, noting that \( \tilde{F}(x, x) = 0 \), the optimality function is defined by

\[
\theta_c(x) \triangleq \min_{h \in \mathbb{R}^n} \tilde{F}(x, x + h).
\]

which can eventually be expressed as

\[
\theta_c(x) = -\min_{\nu \in \Sigma_p, \mu \in \Sigma_q} \left\{ \nu^0 \sum_{j=1}^{q} \mu^j \left[ \psi(x) - f^j(x) + \eta c(x)^+ \right] + \sum_{k=1}^{p} \nu^k [c(x)^+ - c^k(x)] \\
+ \frac{1}{2\delta} \left[ \nu^0 \sum_{j=1}^{q} \mu^j \nabla f^j(x) + \sum_{k=1}^{p} \nu^k \nabla c^k(x) \right]^2 \right\},
\]

(55)

where

\[
\Sigma_p^0 = \left\{ \mu \in \mathbb{R}^{p+1} \mid \sum_{j=0}^{p} \mu^j = 1, \ \mu^j \geq 0, \ j = 0, \ldots, p \right\}
\]

(56)

and the corresponding search direction is given by

\[
h(x) = -\frac{1}{\delta} \left[ \nu^0 \sum_{j=1}^{q} \mu^j \nabla f^j(x) + \sum_{k=1}^{p} \nu^k \nabla c^k(x) \right].
\]

(57)

The search direction \( h(\cdot) \) has the property that it is a descent direction for \( \psi(\cdot) \) at \( x \) when \( c(x) \leq 0 \) and it is a descent direction for \( c(\cdot) \) at \( x \) when \( c(x) \geq 0 \).

From equation (55), if \( \theta_c(x) = 0 \), then

\[
\nu^0 \sum_{j=1}^{q} \mu^j \left[ \psi(x) - f^j(x) + \eta c(x)^+ \right] + \sum_{k=1}^{p} \nu^k [c(x)^+ - c^k(x)] = 0,
\]

(58)

\[
\nu^0 \sum_{j=1}^{q} \mu^j \nabla f^j(x) + \sum_{k=1}^{p} \nu^k \nabla c^k(x) = 0,
\]

(59)

which corresponds to the \textit{generalized Karush-Kuhn-Tucker} conditions when \( \nu^0 > 0 \).
5 Consistent approximations

5.1 The concept

Numerical integration, when used in the solution of an optimization problem involving partial differential equations, results in the replacement of the original optimization problem by a family of discretized optimization problems, parametrized by the integration mesh or step-size. It is essential to ensure that the optimal solutions (resp. stationary points) of the discretized problems converge to optimal solutions (resp. stationary points) of the original problem, as the mesh size goes to zero. This is by no means automatically true. A sufficient condition for the desired effects to be true is that the discretized problems and the linearized problems defining their optimality functions epi-converge to the original problem and to the linearized problem defining its optimality function, respectively. In [6] one may find an abstract formulation of these conditions stated as follows, in a form that is easy to use in applications.

Consider a problem \( P \) of the form

\[
P = \min_{x \in X} f(x),
\]

with its associated discrete problem \( P_N \)

\[
P_N = \min_{x \in X_N} f_N(x).
\]

**Definition 5.1** The sequence of approximating problems \( \{P_N\}_{N \in K} \) epi-converges to the problem \( P \) \((P_N \xrightarrow{\text{Epi}} P)\) if, and only if,

(i) for every \( x \in X \), there exists a sequence \( \{x_N\}_{N \in K} \), with \( x_N \in X_N \), such that \( x_N \xrightarrow{K} x \) as \( N \to \infty \) and \( \lim \inf f_N(x_N) \leq f(x) \),

(ii) for every infinite sequence \( \{x_N\}_{N \in K'} \), \( K' \subset K \), such that \( x_N \in X_N \), \( \forall N \in K' \) and \( x_N \xrightarrow{K'} x \), as \( N \to \infty \), \( x \in X \) and \( \lim f_N(x_N) \geq f(x) \).

**Definition 5.2** Let \( \theta(\cdot), \theta_N(\cdot), N \in \mathbb{N} \), be the optimality functions for the problems \( P \) and \( P_N \), respectively. The pairs \( (P_N, \theta_N) \), in the sequence \( \{(P_N, \theta_N)\}_{N \in \mathbb{N}} \) are said to be consistent approximations to \( (P, \theta) \) if \( P_N \xrightarrow{\text{Epi}} P \) and for any infinite sequence \( \{x_N\}_{N \in K} \), \( K \subset \mathbb{N} \), such that \( x_N \to x \), \( \lim \theta_N(x_N) \leq \theta(x) \), and \( \theta_N(x) < 0 \) for all \( x \in X_N \), \( N \in \mathbb{N} \).
5.2 The unconstrained case

Returning to the unconstrained case of our problem, it will be shown that the pairs \( \{ (P_{u,N}, \theta_{u,N}) \} \) are consistent approximations to the pair \( (P_u, \theta_u) \) defined by (39) and (45).

In view of the results in sections 3.2.1 and 3.2.2, and equations (26), (29), and (35), in particular, one obtains the following result:

**Theorem 5.3**

(i) For all \( j \in \mathbf{q} \), the functions \( f^j(\cdot) \) and \( f^j_N(\cdot) \) are continuous and their gradients \( \nabla f^j(\cdot) \) and \( \nabla f^j_N(\cdot) \) exist and are continuous.

(ii) For all \( j \in \mathbf{q} \) and for all \( x \in \mathbf{X} \)

\[
\lim_{N \to \infty} f^j_N(x) = f^j(x), \quad \lim_{N \to \infty} \nabla f^j_N(x) = \nabla f^j(x).
\]

5.2.1 Epi-convergence

**Theorem 5.4** The problems \( P_{u,N} \) generated using the boundary element method epi-converge to the problem \( P_u \).

**Proof:** Since the functions \( f^j(\cdot) \) and \( f^j_N(\cdot) \) are continuous, the max functions \( \psi(\cdot) \) and \( \psi_N(\cdot) \) are also continuous (see [6, Corollary 5.4.4]). Therefore, for every converging sequence \( x_N \to x \), as \( N \to \infty \), \( \psi(x_N) \to \psi(x) \). It now follows from equation (20) and (26) that there exist constants \( K^j < \infty \) and a strictly decreasing function \( \Delta(N) \) such that

\[
|f^j_N(x) - f^j(x)| \leq K^j \Delta(N), \quad \forall N \in \mathbf{N} \text{ and } \forall j \in \mathbf{q}.
\]

Therefore,

\[
\max_{j \in \mathbf{q}} |f^j_N(x) - f^j(x)| \leq K \Delta(N), \quad \text{with } K = \max_{j \in \mathbf{q}} K^j.
\]

Consequently,

\[
|\max_{j \in \mathbf{q}} f^j_N(x) - \max_{j \in \mathbf{q}} f^j(x)| \leq \max_{j \in \mathbf{q}} |f^j_N(x) - f^j(x)| \leq K \Delta(N),
\]

which implies that \( |\psi_N(x) - \psi(x)| \leq K \Delta(N) \), and hence

\[
\psi_N(x) - \psi(x) \to 0 \quad \text{as} \quad N \to \infty.
\]

Since

\[
\psi_N(x_N) - \psi(x) = [\psi_N(x_N) - \psi(x_N)] + [\psi_N(x) - \psi(x)],
\]
and since the two bracketed terms vanish as \( N \to \infty \), it follows that
\[
\psi_N(x_N) \to \psi(x) \quad \text{as} \quad N \to \infty ,
\]
which proves that the conditions in Definition 5.1 are satisfied. \( \square \)

5.2.2 Optimality functions

It follows from equation (45), that the optimality functions \( \theta_u(\cdot) \) and \( \theta_{u,N}(\cdot) \) can be rewritten in the form
\[
\theta_u(x) = \min_{\xi \in G\psi(x)} \left\{ \xi^0 + \frac{1}{2\delta}||\xi||^2 \right\},
\]
\[
\theta_{u,N}(x) = \min_{\xi \in G\psi_N(x)} \left\{ \xi^0 + \frac{1}{2\delta}||\xi||^2 \right\},
\]
where \( \xi = (\xi^0, \xi) \in \mathbb{R}^{n+1} \), and
\[
G\psi(\cdot) \triangleq \text{co} \left\{ \left( \psi(\cdot) - f^j(\cdot), \nabla f^j(\cdot) \right) \right\}, \quad (72)
\]
\[
G\psi_N(\cdot) \triangleq \text{co} \left\{ \left( \psi_N(\cdot) - f^j_N(\cdot), \nabla f^j_N(\cdot) \right) \right\}. \quad (73)
\]

Concerning continuity of set-valued maps, the Painlevé-Kuratowski sense of convergence is adopted here, see [17]. In this sense, an infinite set of set \( \{A_N\}_{N=1}^{\infty} \) converges to a set \( A \), if \( A \subset \text{lim}A_N \subset \overline{\text{lim}}A_N \subset A \), where \( \text{lim} \) and \( \overline{\text{lim}} \) denote the inner and outer limits of \( A_N \), i.e., the sets of limit and cluster points, respectively, see, e.g., [6, Chapter 5.3.1].

Lemma 5.5 The set-valued maps \( G\psi(\cdot) \) and \( G\psi_N(\cdot) \) are continuous.

Proof: (a) First, consider the function \( G\psi(\cdot) \). To show that it is continuous, one needs to show that \( G\psi(x) \subset \text{lim}G\psi(x_N) \subset \overline{\text{lim}}G\psi(x_N) \subset G\psi(x) \). One may begin by showing that \( \overline{\text{lim}}G\psi(x_N) \subset G\psi(x) \).

Consider a converging sequence \( \bar{\xi}_N \overset{K}{\to} \bar{\xi} \), \( K \subset \mathbb{N} \), such that \( \bar{\xi}_N \in G\psi(x_N) \), for all \( N \). It follows from Caratheodory’s Theorem, that \( \bar{\xi}_N \) can be expressed in the form
\[
\bar{\xi}_N = \sum_{j=1}^{q} \mu^j_N \bar{\xi}_j(x_N) , \quad (74)
\]
where
\[ \xi_j(\cdot) = \begin{pmatrix} \psi(\cdot) - f_j^i(\cdot) \\ \nabla f_j^i(\cdot) \end{pmatrix} \quad \text{and} \quad \mu_N \in \Sigma_q. \tag{75} \]

Since \( f_j^i(\cdot), \nabla f_j^i(\cdot) \) and \( \psi(\cdot) \) are continuous, the function \( \xi_j(\cdot) \) are also continuous for all \( j \in q \). Hence, \( \xi_j(x_N) \to \xi_j(x) \) as \( N \to \infty \). Therefore, since \( \Sigma_q \) is a compact set, there exists a subset \( K' \subset K \subset \mathbb{N} \) such that \( \mu_N \xrightarrow{K'} \mu \), which implies that
\[
\sum_{j=1}^{q} \mu_j^N \xi_j(x_N) \xrightarrow{K'} \sum_{j=1}^{q} \mu_j^N \xi_j(x). \tag{76}
\]

By inspection, \( \sum_{j=1}^{q} \mu_j^N \xi_j \in \mathcal{C}(\psi(x)) \), and therefore \( \xi_N \xrightarrow{K'} \bar{\xi} \in \mathcal{C}(\psi(x)) \), which implies that \( \lim \mathcal{C}(\psi(x_N)) \subset \mathcal{C}(\psi(x)) \).

Next it will be shown that \( \bar{\mathcal{C}}(\psi(x)) \subset \lim \mathcal{C}(\psi(x_N)) \).
Let \( \bar{\xi} \in \mathcal{C}(\psi(x)) \) be arbitrary. Then, by Theorem 7.5, there exists a \( \mu \in \Sigma_q \), such that \( \bar{\xi} = \sum_{j=1}^{q} \mu_j^N \xi_j(x) \). Next, let the infinite sequence \( \{\xi_N\}_{N \in K}, K \subset \mathbb{N} \), be defined by \( \xi_N = \sum_{j=1}^{q} \mu_j^{N,N} \xi_j(x) \), obviously \( \xi_N \in \mathcal{C}(\psi(x_N)) \). Since by assumption, \( x_N \to x \), as \( N \to \infty \), it follows by continuity that \( \xi_N \to \bar{\xi} \), as \( N \to \infty \), which implies that \( \bar{\mathcal{C}}(\psi(x)) \subset \lim \mathcal{C}(\psi(x_N)) \).

(b) The proof that the maps \( \bar{\mathcal{C}}(\psi_N(\cdot)) \) are continuous is identical to the one above and is therefore omitted.

Since the set-valued maps \( \bar{\mathcal{C}}(\psi(\cdot)) \) and \( \bar{\mathcal{C}}(\psi_N(\cdot)) \) are continuous, one obtains the following result from [6, Corollary 5.4.2]:

**Lemma 5.6** The optimality functions \( \theta_u(\cdot) \) and \( \theta_{u,N}(\cdot) \), defined by (71), are continuous.

**Lemma 5.7** Let the set-valued maps \( \bar{\mathcal{C}}(\psi(\cdot)) \) and \( \bar{\mathcal{C}}(\psi_N(\cdot)) \) be associated with the original problem and the boundary element model, respectively, and defined as above. If \( \{x_N\}_{N=0}^{\infty} \) is an infinite sequence such that \( x_N \to x \), as \( N \to \infty \), then,
\[
\bar{\mathcal{C}}(\psi_N(x_N)) \to \bar{\mathcal{C}}(\psi(x)). \tag{77}
\]

The proof of the Lemma 5.7 can be carried out by using arguments very similar to those used for Lemma 5.5, and hence is omitted.

**Lemma 5.8** Let \( \theta_u(\cdot), \theta_{u,N}(\cdot), N \in \mathbb{N}, \) be the optimality functions for the problems \( \mathcal{P}_u \) and \( \mathcal{P}_{u,N} \) respectively. Then for any infinite sequence \( \{x_N\}_{N \in K}, K \subset \mathbb{N}, \) such that \( x_N \to x \),
\[
\lim \theta_{u,N}(x_N) \leq \theta_u(x). \tag{78}
\]
Consistent approximations for boundary elements in shape optimization with uncertainty

**Proof:** Since by Lemma 5.7, $G_N(x_N) \rightarrow G(x)$ as $N \rightarrow \infty$,

$$
\lim_{N \rightarrow \infty} \theta_{u,N}(x_N) = \lim_{N \rightarrow \infty} \left\{ - \min_{\xi \in G_N(x_N)} \left\{ \xi^0 + \frac{1}{2\delta} ||\xi||^2 \right\} \right\} = \theta_u(x),
$$

(79)

which obviously implies (78).

In view of the above Lemmas, the following result is obvious.

**Theorem 5.9** The pairs $\{(P_{u,N}, \theta_{u,N})\}_{N \in \mathbb{N}}$ generated by the boundary element method and defined by equations (61) and (71), with $f_N(\cdot) \triangleq \psi_N(\cdot)$ and $X_N = \mathbb{R}^n$, are consistent approximations to $(P_u, \theta_u)$.

5.3 The constrained case

**Assumption 5.10** The constraint set is independent of the discretization parameter, i.e., for all $N$, $X_N \triangleq X = \{x \in \mathbb{R}^n | c_k(x) \leq 0, \ \forall \ k \in \mathbb{P} \}$.

5.3.1 Epi-convergence

In the case where $X_N = X$, in view of Theorem 5.4, the conditions for epi-convergence are trivially satisfied.

5.3.2 Optimality functions

For the problem with constraints, the optimality functions can be expressed in the compact form

$$
\theta_c(x) = - \min_{\xi \in GF(x)} \left\{ \xi^0 + \frac{1}{2\delta} ||\xi||^2 \right\},
$$

(80)

$$
\theta_{c,N}(x) = - \min_{\xi \in GF_N(x)} \left\{ \xi^0 + \frac{1}{2\delta} ||\xi||^2 \right\},
$$

(81)

where $GF(\cdot)$ is deduced from the expression for $\theta_c(\cdot)$ in equation (55), i.e.,

$$
GF(\cdot) \triangleq \text{co}\left\{ \text{co}\left\{ \left( \psi(\cdot) - f^j(\cdot) + \eta c(\cdot) + \nabla f^j(\cdot) \right) \right\} \setminus \text{co}\left\{ \left( c(\cdot) + c^k(\cdot) + \nabla c^k(\cdot) \right) \right\} \right\},
$$

(82)

and $GF_N(\cdot)$ is obtained by replacing $f^j(\cdot)$ and $\psi(\cdot)$, in (82) $f^j_N(\cdot)$ and $\psi_N(\cdot)$, respectively.

**Lemma 5.11** Consider that Assumptions 5.3 and 5.10 are satisfied. For any infinite sequence $\{x_N\}_{N \in \mathbb{N}}$, $K \subset \mathbb{N}$, such that $x_N \rightarrow x$

$$
\lim_{N \rightarrow \infty} \theta_{c,N}(x_N) = \theta_c(x).
$$

(83)
Proof: It can be shown in exactly the same way as for the unconstrained case that for any infinite sequence \( \{x_N\}_{N \in \mathbb{N}} \) converging to a point \( x \), \( GF_N(x_N) \to GF(x) \) as \( N \to \infty \). It now follows immediately from [6] that \( \theta_{c,N}(x_N) \to \theta_c(x) \), which implies (83).

At this point, our main result follows directly:

**Theorem 5.12** The pairs \( \{(P_{c,N}, \theta_{c,N})\}_{N \in \mathbb{N}} \) generated by the boundary element method and defined by equations (61) and (81) with \( X_N = X \), are consistent approximations to \( (P_c, \theta_c) \).

### 6 Example and numerical aspects

#### 6.1 A simple minimax problem

In this section, a minimax problem is formulated to account for the presence of uncertainty in the shape optimization of the two-dimensional square plate with circular holes in Figure 1(a). For this particular system,

\[
t = 1 \quad \text{on } \Gamma_r ,
\]

as shown in Figure 1(a). All holes are subject to homogeneous Neumann boundary conditions. The remaining boundary conditions are illustrated in the same figure.

The position of two of the three circular holes in the preceding system is fully prescribed without any uncertainty. The objective is to find the optimal position of the third hole when its placement is uncertain. Since the goal is to minimize the worst-case performance, the problem is expressed in the minimax form (1). The uncertainty is introduced in the form of a perturbation to the design variable, such that \( f^j(x) \triangleq f(x + y_j) \). With reference to Figures 1(b,c), the set \( \{y_j\}_{j=1}^q \) is chosen such that \( \{x + y_j\}_{j=1}^q \) are the points at the corners and at the center of an uncertainty box, hence \( q = 5 \). Furthermore, constraint functions \( c^k(\cdot) \) are defined to prevent the holes from overlapping or reaching any boundary. In the absence of uncertainty, the deterministic problem is identified with \( \psi(x) = f^1(x) \triangleq f(x) \), i.e., only the point at the center of the uncertainty box.

In the ensuing simulations, the elastic properties are set to \( \lambda = \mu = 4 \), the plate is of dimension \( 100 \times 100 \), all holes have diameter of 10 units, and the uncertainty box is of dimensions \( 2 \times 2 \) (“small”) and \( 10 \times 10 \) (“large”). All simulations are performed using quadratic boundary elements. First, consider the case of a fixed uniform discretization with 8 nodes on the boundary of each hole and 24 nodes on each side of the exterior boundary.
In Figure 1(b,c), one sees that for a sufficiently small range of uncertainty, the algorithm converges to a solution close to that of the deterministic problem which corresponds to the configuration where the moving hole is aligned with one of the fixed ones. For a larger range of uncertainty values, the solution is significantly different. This is due to the fact that the cost function of the deterministic problem is very sensitive to perturbation around its minimum, see Figure 2(b). Also, Figures 1(b,c) show that the introduction of uncertainty modifies the topology of the cost function. Indeed, even if the cost function of the deterministic problem is convex within $\Omega_x$, this is not the case in the presence of uncertainty as this is clearly illustrated in the case of large uncertainty where the solution depends crucially on the initial placement of the hole.

6.2 Some implementational aspects of the boundary element method

Using the boundary element method, each function $f_j^N(x)$, associated with the discretization parameter $N$ is computed after solving the algebraic system of the form $A_j u_N = b_j$, see equation (16), where it is understood that $A_j = A(x, y_j)$ and $b_j = b(x, y_j)$. The use of boundary elements has the advantage that only the rows and columns of $A_j$ affected by the value of the uncertainty parameter $y_j$, differ for two different values of $j$. Therefore, once $A_j$ has been computed, only the elements of $A_j$ affected by $y_{j+1}$ have to be updated to obtain $A_{j+1}$. In addition, note that the part of $A_j$ which is not associated with the uncertainty or design variables needs to be computed only once.

Table 1 shows that exploiting this property leads to a substantial reduction of the total computing time. This is especially true when the uncertainty affects only a portion of the boundary. Also, if only few elements of $A_j$ change from one value of $j$ to another, the linear algebraic systems can be solved efficiently using iterative methods starting with an initial guess that coincides to a previously computed solution. This remark is relevant to problems with large number of unknowns where iterative methods may be more efficient than direct methods.

6.3 Consistent approximations and the boundary element method

Having proved that the pairs $\{(P_N, \theta_N)\}_{N \in \mathbb{N}}$ generated by the boundary element method are consistent approximations to $(P, \theta)$, one may proceed to solve the minimax problem using the adaptive approximation algorithms ALG1 and ALG2 described in the Appendix. Using these algorithms ensures that every accumulation point $\hat{x}$ of $\{x_N\}_{N=0}^\infty$ is a feasible
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<table>
<thead>
<tr>
<th>N1</th>
<th>N2</th>
<th>T1</th>
<th>T2</th>
<th>T3</th>
</tr>
</thead>
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<td>120</td>
<td>8</td>
<td>7.5</td>
<td>1.1</td>
<td>&lt; 0.01</td>
</tr>
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<td>180</td>
<td>12</td>
<td>16.4</td>
<td>2.2</td>
<td>&lt; 0.01</td>
</tr>
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<td>240</td>
<td>16</td>
<td>28.5</td>
<td>3.9</td>
<td>&lt; 0.01</td>
</tr>
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<td>300</td>
<td>20</td>
<td>44.5</td>
<td>6.1</td>
<td>&lt; 0.01</td>
</tr>
<tr>
<td>360</td>
<td>24</td>
<td>64.0</td>
<td>8.8</td>
<td>&lt; 0.01</td>
</tr>
</tbody>
</table>

Table 1: Compute time (in sec) to solve a boundary element system. N1: total number of nodes, N2: number of nodes affected by the change of geometry, T1: time to compute $A_j$, T2: time to compute $A_{j+1}$ from $A_j$, T3: time to evaluate $f^j(u_N)$.

point and satisfies $\theta(\mathbf{x}) = 0$.

Figure 2 illustrates the fact that the cost function values $f_N(x)$ can vary significantly for different values of $N$. However, this does not affect the convergence properties of the algorithm which uses a coarse mesh when far from a solution and refines the mesh adaptively as a solution is approached, resulting in considerable savings in computing time over setting the mesh to the required ultimate fineness and then using an algorithm, such as the PPP Algorithm (see the Appendix), with the fine mesh during the entire computation. In practice, a rule is introduced to signify convergence. This can be based on the convergence of the solution in an appropriate norm, a maximum value for $N$ or a combination of both.

Consider next the adaptive algorithms ALG1 and ALG2. First, refer to the refinement condition, given by equation (95) for the unconstrained problem, and (97) for the constrained problem. These tests compare the decrease of the cost function for a given $N$ with the value of a strictly monotone decreasing function $\Delta : \mathbb{N} \rightarrow \mathbb{R}$ such that, for $K$ a finite constant, for all $N \geq N_0$ and for all $x \in X$,

$$|\psi_N(x) - \psi(x)| \leq K\Delta(N).$$

(85)

Although the value of $\tau$ in ALG1 and ALG2 can be specified arbitrarily, a reasonable choice is to make it proportional to the decrease of the cost function, namely

$$\tau = \frac{1}{M} \frac{|\psi_N(x_1) - \psi_N(x_0)|}{\Delta(N_0)^\omega},$$

(86)

where $x_1 = I_u(x_0)$ for the unconstrained case, see equation (90) in the Appendix, and

$$\tau = \frac{1}{M} \frac{F(x_0, x_1)}{\Delta(N_0)^\omega},$$

(87)
where $x_1 = I_c(x_0)$ for the constrained case, see equation (93) in the Appendix.

Figure 5 shows the influence of the value of $M$ on the discretization for the problem represented on Figure 4. A large value for $M$ requires the decrease of the cost function to be small for a mesh refinement to occur. This may force the algorithm to come unnecessarily close to the solution of problem $P_{N_i}$ before increasing the discretization parameter and move to the problem $P_{N_{i+1}}$. On the other hand, a small value of $M$ may make the tests (95) and (97) fail too quickly, thus leading to a rapid mesh refinement and unnecessarily increasing the required computational work. On the other hand, $\omega$ can always be set very close to unity, e.g., $\omega = 0.99$.

For the problem represented in Figure 4, it was estimated that for $M = 10$, the adaptive approximation algorithm was approximately 2.5 faster than performing the entire optimization with the fine mesh. However, it should be noted that time estimate is code-dependant and may vary with the boundary-element implementation.

To the original adaptive approximation algorithms, see [6, Section 3.3], one may add the option for the value of the discretization parameter $N$ to decrease under the condition that the value of the cost function decreases enough with respect to $\tau \Delta(N)$. In analogy to the refinement test, a derefinement test is also proposed and implemented here. The objective of derefinement is to accelerate the convergence when the rate of decrease of the cost is becoming progressively faster, see, e.g., Figure 3. The derefinement condition is

$$-\tau \Delta(N_i)^{2\omega} \leq \psi_{N_i}(x_{i+1}) - \psi_{N_i}(x_i),$$  \hspace{1cm} (88)$$

for the problem $P_u$ and

$$-\tau \Delta(N_i)^{2\omega} \leq F_{N_i}(x_i, x_{i+1}),$$  \hspace{1cm} (89)$$

for the problem $P_c$, such that if (88) or (89) is satisfied, one may remove the restriction that the discretization parameter has to increase. It should be observed that this rule is allowed by the original algorithm as long as the condition $N_i \geq N_{i-1}$ is enforced after a given number of iterations $I$. The idea behind mesh derefinement lies in the fact that far from the solution almost any optimization technique can be employed and the refinement tests (95, 97) can be safely ignored.

Figure 3 presents the effect of the evolution of the discretization during the optimization of the problem associated with Figure 1. After just a few iterations, the decrease of the cost function is such that a first refinement is required by the algorithm. As the algorithm continues, the reduction of the cost function increases in such a way that a derefinement is
then allowed until the solution eventually converges to the optimal solution with the model with the desired discretization.

The preceding procedure can be followed for any cost function for which it is possible to obtain a sequence of pairs $\{(P_N, \theta_N)\}_{N \in \mathbb{N}}$ which are consistent approximations to the pair $(P, \theta)$ of the original problem.

7 Conclusion

The concept of consistent approximations is applicable to the boundary element analysis of shape optimization problems for linearly elastic systems. This concept results in adaptive optimization algorithms that yield significant savings in computing time. These savings stem from the use of coarse meshes for the early steps of the optimization and the progressive use of finer meshes as the iterate converges to a point that the adaptive approximation algorithm guarantees to be an optimal solution.

References


**Appendix: Algorithms**

**PPP algorithm**

For the unconstrained case, the iteration function $I_u : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by the PPP algorithm, see [6, Algorithm 2.4.1.], with the parameters $\alpha \in (0, 1)$, $\beta \in (0, 1)$, $\delta > 0$ and has the following form

$$I_u(x) \triangleq x + \nu(x)h(x)$$

where $h(\cdot)$ is defined by (47) and the $\nu(x)$ is given by the Armijo step-size rule,

$$\nu(x) = \max_{k \in \mathbb{N}} \left\{ \beta^k \mid \psi(x_i + \beta^k h(x_i)) - \psi(x_i) - \beta^k \alpha \theta_u(x_i) \leq 0 \right\}.$$  \hspace{1cm} (91)

The choice of parameters $\alpha$, $\beta$ and $\delta$ influence the behavior of the algorithm, as follows. The parameter $\beta$ governs the step-size grid explored by the step-size rule. For convex problems, setting $\alpha = 1$, give the best rate of convergence. For non-convex problems, a smaller value
might be preferable. Also, for the best rate of convergence, $\delta$ should be bracketed by the smallest and largest eigenvalues of any second derivative matrix $f_{jxx}(x)$. A very simple way to ensure this, see [6], is to choose $\delta$ to be the average of the $\delta_j$ such that

$$\delta_j = \frac{f_j(x_1) - f_j(x_0) - \langle \nabla f_j(x_0), x_1 - x_0 \rangle}{||x_1 - x_0||^2}. \quad (92)$$

**Polak-He algorithm**

For the constrained case, the iteration function $I_c : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined by the Polak-He algorithm, see [6, Algorithm 2.6.1.], as follows

$$I_c(x) \triangleq x + \nu(x)h(x) \quad (93)$$

where $h(\cdot)$ is defined by (57) and the $\nu(x)$ is given by

$$\nu(x) = \max_{k \in \mathbb{N}} \left\{ \beta^k \mid F(x_i, x_i + \beta^k h(x_i)) - \beta^k \alpha \theta_u(x) \leq 0 \right\}. \quad (94)$$

**Adaptive approximation algorithm ALG1 (the unconstrained case)**

Step 0. Given $I$, $x_0 \in X_{N-1}$, $\omega \in (0, 1)$, $\tau > 0$, see (86), and data $N_{-1} \in \mathbb{N}$, set $i = 0$.

Step 1. If $i \leq I$ and $-\tau \Delta(N_{i-1})^{2\omega} \leq \psi_{N_{i-1}}(x_{i+1}) - \psi_{N_{i-1}}(x_i)$, where $x_{i+1} = I_u(x_i)$, compute the smallest $N_i$ such that

$$\psi_{N_i}(x_{i+1}) - \psi_{N_i}(x_i) \leq -\tau \Delta(N_i)^\omega \quad (95)$$

Else, compute the smallest $N_i$ such that $N_i \geq N_{i-1}$, $x_{i+1} = I_u(x_i)$ and

$$\psi_{N_i}(x_{i+1}) - \psi_{N_i}(x_i) \leq -\tau \Delta(N_i)^\omega \quad (96)$$

Step 2. Replace $i$ by $i + 1$, go to Step 1.

**Adaptive approximation algorithm ALG2 (the constrained case)**

Step 0. Given $I$, $x_0 \in X_{N-1}$, $\omega \in (0, 1)$, $\tau > 0$, see (87), and data $N_{-1} \in \mathbb{N}$, set $i = 0$.

Step 1. If $i \leq I$ and $-\tau \Delta(N_{i-1})^{2\omega} \leq F_{N_{i-1}}(x_i, x_{i+1})$, where $x_{i+1} = I_c(x_i)$, compute the smallest $N_i$ such that

$$F_{N_i}(x_i, x_{i+1}) \leq -\tau \Delta(N_i)^\omega \quad (97)$$

Else, compute the smallest $N_i$ such that $N_i \geq N_{i-1}$, $x_{i+1} = I_c(x_i)$ and

$$F_{N_i}(x_i, x_{i+1}) \leq -\tau \Delta(N_i)^\omega \quad (98)$$

Step 2. Replace $i$ by $i + 1$, go to Step 1.
Figure 1: Canonical problem and contour plots of the cost function in the domain $\Omega_x$. The boxes represent the ranges of uncertainty in the positioning of the moving hole: (a) Boundary conditions (zero displacement in the horizontal direction on $\Gamma_l$, uniform traction in the horizontal direction on $\Gamma_r$) and hole configuration with range of free hole placement, (b,c) convergence of the minimax problem for “large” and “small” uncertainty, respectively black and white box, starting from two initial hole placements $x_0$. 

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Figure 2: Three-dimensional plot of the cost (i.e., the square of the 2-norm of the horizontal displacement along \( \Gamma_r \)) as a function of the position of the hole within the domain \( \Omega_x \) depicted in Figure 1: (a) 48-node discretization, (b) 96-node discretization
Figure 3: (a) Evolution of the discretization and (b) evolution of the cost function using the adaptive approximation algorithm ALG2 (with derefinement option) for the system presented in Figure 1.
Figure 4: (a) Initial and (b) optimal designs with the representation of the evolution of the design using the adaptive approximation algorithm ALG2 for two values of the parameter $M$. The solid line corresponds to $M = 10$ and the dashed line to $M = 50$. 
Figure 5: (a) Evolution of the discretization and (b) evolution of the cost function using ALG2 for the system presented in Figure 4 for two values of the parameter $M$. The solid line corresponds to $M = 10$ and the dashed line to $M = 50$. 