Material Growth in Thermoelastic Continua: Theory, Algorithmics, and Simulation

Chet VIGNES and Panayiotis PAPADOPoulos
Department of Mechanical Engineering
University of California, Berkeley, CA 94720-1740, USA

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Abstract

A theoretical formulation of stress-induced volumetric material growth in thermoelastic continua is developed. Material growth is regulated by a three-surface activation criterion and corresponding flow rules. A simple model is proposed based on this formulation and its algorithmic implementation, including a method for solving the underlying differential/algebraic equations for growth, is examined in the context of an implicit finite element method. Selected numerical simulations are presented that showcase the predictive capacity of the model for both soft and hard biomaterials.

Keywords: Growth; continuum modeling; constitutive modeling; activation surfaces; finite element method.
1 Introduction

Biomaterials are distinguished from traditional engineering materials by their capacity to not only deform, but also to develop and adapt in response to external stimuli. Development can occur by growth, remodeling, or morphogenesis. Growth (respectively, resorption) is the ability of the biomaterial to gain (respectively, lose) mass. Remodeling is the rearrangement of biomaterial microstructure to alter material properties or material symmetry. Morphogenesis refers to a combination of growth and remodeling to generate form and typically applies to embryonic development. Functional adaptation refers to the capacity of the biomaterial to change its configuration and material properties in response to altered physical stimuli. Adaptation can be effected by both growth and remodeling. Further detail on the biological concepts and nomenclature of biomaterial development can be found in the review article by Taber [1].

Material growth is known to be influenced by mechanical factors; numerous examples appear in the actively researched biomechanics disciplines of bone mechanics, vascular mechanics, and tumor bioscience. In bone mechanics, Wolff’s law, perhaps the most familiar example of functional adaptation of biomaterial to mechanical stimuli, models trabecular bone patterns aligned with principal stress trajectories [2]. Although the specific law has been largely disproved [3, 4, 5], the phenomenon of functional adaptation is generally accepted. In the study of arteries, experiments have shown that an increase in blood pressure triggers arterial wall thickening and accumulation of residual stress [6, 7]. An important outcome of these experiments is the observation that not only does stress modulate the growth process, but also growth alters the stress distribution; growth and stress response are coupled. In the study of tumor development, Helmlinger et al. [8] showed that stress inhibits tumor growth in vitro. In this work, multicellular tumor spheroids were embedded and cultured in agarose gels of variable stiffness. The spheroids grew larger in less stiff gels; however, the interface stress at which tumor growth was inhibited was similar across all agarose gels. The study supports the concept of a growth-inhibiting threshold stress. It should be noted that there is no general agreement among researchers that it is stress rather than other mechanical factors (e.g., strain, strain rate, strain-energy) that modulates growth. Experimental evidence can be found to support each claim. A detailed account of the opposing arguments and a resolving viewpoint is presented by Humphrey [9], which concludes that any convenient continuum metric capable of being correlated to growth for the problem of interest is valid.
The intent of this paper is to model stress-induced growth processes at the continuum level. A brief review of continuum scale modeling efforts to date follows. In an initial effort to cast growth concepts into the mathematical framework of continuum mechanics, Hsu [15] revised the standard balance of mass equation to include mass sources in the interior of the body. The remaining balance laws were assumed to be unaltered and representative results for stress-dependent growth in linear elastic materials undergoing homogeneous deformation were obtained. The notion of biomaterials as materials with fading memory was also introduced, although the memory time scale was neglected in the examples. It was recognized that the time scales of growth and loading differ by orders of magnitude and must receive careful treatment in modeling. The theory of growth was extended to a finite deformation, thermomechanical continuum base for bone remodeling by Cowin and Hagedus [16]. In this work, a closed two-phase (solid-fluid) system was developed. The fluid phase was considered only in its capacity to transfer mass, momentum, and energy to the load-bearing solid phase capable of growth. Lubarda and Heger [17], Epstein and Maugin [18], and Kuhl and Steinmann [19] replaced the multiphase model with a single constituent model with irreversible sources and fluxes of momentum, energy, and entropy.

Formulating the kinematics of growth for finite deformations, Skalak [20, 21] recognized the interaction between growth strain and residual stress. If an arbitrary deformation-induced growth process causes an initially stress-free body to develop residual stresses, the growth strain field cannot be compatible. This observation motivated Rodriguez et al. [22] to employ a multiplicative decomposition of the total deformation gradient into a growth deformation tensor and an elastic deformation tensor analogous to the decomposition used in finite plasticity models as suggested by Kröner [23] and Lee [24]. Growth models utilizing this decomposition are predominant in the literature [25, 26, 27], although alternative theories have been suggested [28].

The present work contributes a broad constitutive and computational framework for the study of growth in both hard and soft biomaterials at a macroscopic scale. In the theory of growth mechanics, it is possible to interpret growth as change in mass due to either density change at fixed volume or volume change at fixed density. The former, commonly assumed for hard materials, is constitutively determined while the latter, commonly assumed for soft materials, is kinematically-driven. The framework developed in this article permits a general growth process with simultaneous density and volume changes to investigate coupled constitutive and kinematic response; the special cases for hard and soft materials can be recovered trivially. A three-surface growth activation criterion is proposed to account
for the three possible states of biological homeostasis, growth, and resorption. The homeostatic state allows for ranges of applied loading which produce no growth response. The activation criterion and its associated constraints are described by a differential/algebraic system. Computationally, an algorithmic implementation for this system, which is distinct for the cases of growth and resorption due to stability issues, is developed within an implicit finite element method. The model is rate-independent; applied loading is considered to be time-averaged over the characteristic period of growth influence.

The organization of this article adheres to the following format: In Section 2, the balance laws of continuum mechanics are rederived without the conventional assumption of conservation of mass. Section 3 begins by introducing a kinematic assumption for the deformation gradient with particular emphasis on the resulting separation of growth processes into kinematic and constitutive sources. Next, general constitutive relations are assumed and finally an entropy inequality is proposed with resulting thermodynamic constitutive restrictions. In Section 4, a growth activation criterion is formulated. A simple constitutive model of rate-independent finite growth and an implicit integration scheme for the resulting differential/algebraic equations of growth parameter evolution are proposed in Section 5. Selected numerical simulations covering both soft and hard biomaterials are presented in Section 6. Section 7 offers concluding remarks.

2 Balance Laws

Let $X$ denote a material point of a body $\mathcal{B}$ and let the body occupy a region in three-dimensional Euclidean point space $\mathcal{E}^3$. For convenience, select an occupiable reference configuration $\mathcal{R}_\kappa$ of the body $\mathcal{B}$ at time $t_0$ defined by $\kappa : \mathcal{B} \mapsto \mathcal{R}_\kappa \subset \mathcal{E}^3$, where $X = \kappa(X)$ is the position vector of $X$ relative to a fixed origin $O$. Also, let $\mathcal{R}$ be the current configuration of the body at time $t$ defined by $\bar{\chi} : \mathcal{B} \mapsto \mathcal{R} \subset \mathcal{E}^3$, where $x = \bar{\chi}(X,t)$ is the position vector of $X$ relative to an origin $o$. The functions $\kappa$ and $\bar{\chi}$ are assumed to be one-to-one. Under this assumption, $\mathcal{R}$ is the image of $\mathcal{R}_\kappa$ at time $t$ as described by the deformation mapping $\chi_\kappa = \bar{\chi} \circ \kappa^{-1} : \mathcal{R}_\kappa \mapsto \mathcal{R}$, $x = \bar{\chi}(\kappa^{-1}(X,t)) = \chi_\kappa(X,t)$, as in Figure 1. The deformation gradient at a material point of $\mathcal{R}$ relative to $\mathcal{R}_\kappa$ is defined as $F = \text{Grad} \chi_\kappa$. To obtain physical deformations, it is assumed that $F(X,t)$ is invertible for any fixed time $t$, or equivalently, $J = \text{det}(F(X,t)) \neq 0$ for all $(X,t)$.

In the following, the balance laws that govern the motion of a thermomechanical-growth material are reviewed along the lines of earlier development in [18].
2.1 Mass balance

The mass contained in an arbitrary material region $\mathcal{P} \subset \mathcal{R}$ of the current configuration is defined as

$$\mathcal{M}(\mathcal{P}) = \int_{\mathcal{P}} \rho \, dv,$$  \hspace{1cm} (2.1.1)

where $\rho = \rho(x,t)$ is the current mass density. Similarly, the mass contained in the corresponding material region $\mathcal{P}_\kappa \subset \mathcal{R}_\kappa$ of the reference configuration is defined in terms of the referential mass density $\rho_\kappa = \rho_\kappa(X,t)$ as

$$\mathcal{M}(\mathcal{P}_\kappa) = \int_{\mathcal{P}_\kappa} \rho_\kappa \, dV.$$  \hspace{1cm} (2.1.2)

The mass of a material region of the body at a given time $t$ is independent of the particular configuration selected to describe it. Therefore, equating (2.1.2) to (2.1.1), recalling the volume relation $dv = J dV$, and invoking the localization theorem yields

$$\rho_\kappa = \rho J.$$  \hspace{1cm} (2.1.3)

Note that $\rho_\kappa(X,t)$ depends explicitly on time; indeed, it is the pull-back of the spatial density to the reference configuration with time dependence introduced to allow for changes in body mass. Let $\rho_0(X) = \rho_\kappa(X,t_0)$ be the initial referential mass density (at time $t_0$), which corresponds with the classical definition. In the special case where there are no body mass changes between times $t_0$ and $t$, it follows that $\rho_\kappa(X,t) = \rho_\kappa(X,t_0) = \rho_0(X)$.

The rate of change of mass can be non-zero due to volumetric mass sources and mass fluxes, namely

$$\dot{\mathcal{M}}(\mathcal{P}) = \frac{d}{dt} \int_{\mathcal{P}} \rho(x,t) \, dv = \int_{\partial \mathcal{P}} \rho \Gamma \, da + \int_{\partial \mathcal{P}} m \, da,$$  \hspace{1cm} (2.1.4)

where $\Gamma$ is the rate of mass growth per unit mass and $m$ is the mass flux into $\mathcal{P}$ per unit surface area through the boundary $\partial \mathcal{P}$. Using Reynolds’ transport formula, it is readily concluded from (2.1.4) that

$$\int_{\mathcal{P}} (\dot{\rho} + \rho \text{div} \, \mathbf{v} - \rho \Gamma) \, dv = \int_{\partial \mathcal{P}} m \, da,$$  \hspace{1cm} (2.1.5)

where $\mathbf{v} = \dot{x}$ is the velocity of the body. Applying to (2.1.5) the procedure in [29, Theorem IV], it is seen that the mass flux at $(x,t)$ depends only on the outward normal $\mathbf{n}$ to $\partial \mathcal{P}$, namely

$$m = m(x,t; \mathbf{n}).$$  \hspace{1cm} (2.1.6)

Assuming $\int_{\mathcal{P}} |(\dot{\rho} + \rho \text{div} \, \mathbf{v} - \rho \Gamma)| \, dv$ is bounded and employing Cauchy’s tetrahedron argument on (2.1.5) implies that $m$ is linear in $\mathbf{n}$ and that there exists a vector $\mathbf{m}(x,t)$ on
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\( \partial \mathcal{P} \), such that \( m = m(x, t; n) = -m(x, t) \cdot n \). Thus, the divergence theorem produces the relation

\[
\int_{\partial \mathcal{P}} m \, da = - \int_{\mathcal{P}} \text{div} \, m \, dv. \tag{2.1.7}
\]

Substituting (2.1.7) into (2.1.5) and invoking the localization theorem yields the local form of the spatial mass balance equation

\[
\dot{\rho} + \rho \text{div} \, v = \rho \Gamma - \text{div} \, m. \tag{2.1.8}
\]

An equivalent referential form can be derived by multiplying (2.1.8) by \( J \) and using the identity \( J \text{div} \, m = \text{Div}[\,(F^*)^T \cdot m] \), where \( F^* = JF^{-T} \) is the cofactor of \( F \). This leads to

\[
\dot{\rho}_c = \rho_c \Gamma - \text{Div} \, M, \tag{2.1.9}
\]

where \( M = JF^{-1} \cdot m \) is the referential mass flux vector.

### 2.2 Linear and angular momentum balance

The balance law of linear momentum accounting for growth is assumed to take the form

\[
\frac{d}{dt} \int_{\mathcal{P}} \rho v \, dv = \int_{\mathcal{P}} \rho b \, dv + \int_{\partial \mathcal{P}} t \, da + \int_{\mathcal{P}} (\rho \Gamma) \nu \, dv - \int_{\partial \mathcal{P}} (m \cdot n) \nu \, da,
\]

where \( b \) is the body force per unit mass, \( t \) is the traction per unit surface area acting on the boundary \( \partial \mathcal{P} \), and \( \nu \) is the velocity of the new mass entering the body. On the right-hand side of (2.2.1), the first and second terms represent linear momentum changes due to standard applied body forces and tractions, while the third and fourth terms account for changes in linear momentum accompanying the additional mass through volumetric sources and mass fluxes, respectively.

An equivalent form of the balance of linear momentum, consistent with the development in [18], can be derived by setting \( \nu = (\nu - v) + v \) in (2.2.1). This substitution yields

\[
\frac{d}{dt} \int_{\mathcal{P}} \rho v \, dv = \int_{\mathcal{P}} \rho b \, dv + \int_{\partial \mathcal{P}} t \, da + \int_{\mathcal{P}} (\rho \Gamma) \nu \, dv - \int_{\partial \mathcal{P}} (m \cdot n) \nu \, da + \int_{\mathcal{P}} \rho \tilde{b} \, dv + \int_{\partial \mathcal{P}} \tilde{t} \, da,
\]

where

\[
\tilde{b} = \Gamma(\nu - v), \quad \tilde{t} = (m \cdot n)(\nu - v). \tag{2.2.3}
\]

In this representation, \( \tilde{b} \) is interpreted as a body force per unit mass corresponding to irreversible momentum changes from volumetric sources, and \( \tilde{t} \) is interpreted as a traction per unit surface area corresponding to irreversible momentum changes from surface fluxes.
Now, on the right-hand side of (2.2.2), the third and fourth terms account for changes in linear momentum due to the new mass entering the body with the same velocity as the body, and the fifth and six terms reflect irreversible momentum changes from volumetric and surface sources, respectively.

Starting from (2.2.2), the use of Reynolds’ transport formula and the mass balance equation (2.1.8) give rise to

\[
\int_P \rho \dot{v} \, dv = \int_P \rho \mathbf{b} \, dv + \int_{\partial P} \mathbf{t} \, da + \int_P [(\text{div} \mathbf{m}) \mathbf{v} - \text{div} (\mathbf{v} \otimes \mathbf{m})] \, dv + \int_P \rho \tilde{\mathbf{b}} \, dv + \int_{\partial P} \tilde{\mathbf{t}} \, da.
\]

(2.2.4)

As with the standard traction vector \( \mathbf{t} \), the vector \( \tilde{\mathbf{t}} \) depends on the normal vector to the surface at a point and takes the form \( \tilde{\mathbf{t}}(\mathbf{x}, t; \mathbf{n}) = \tilde{T}(\mathbf{x}, t) \mathbf{n} \), where \( \tilde{T} = (\tilde{\mathbf{v}} - \mathbf{v}) \otimes \mathbf{m} \) is a second-order tensor. Then, application of the divergence theorem to (2.2.4) in conjunction with the localization theorem results in the local form of the spatial linear momentum balance equation

\[
\rho \dot{\mathbf{v}} = \rho \mathbf{b} + \text{div} \tilde{T},
\]

(2.2.5)

where

\[
\tilde{\mathbf{b}} = \mathbf{b} + \tilde{\mathbf{b}} + \frac{1}{\rho} (\text{div} \mathbf{m}) \mathbf{v}, \quad \tilde{T} = \mathbf{T} - \mathbf{v} \otimes \mathbf{m} + \tilde{T},
\]

(2.2.6)

are the “effective” body force and “effective” Cauchy stress, respectively. Note that \( \mathbf{T} \) in (2.2.6)\(_2\) is the standard Cauchy stress tensor related to the traction vector \( \mathbf{t} \) according to \( \mathbf{t} = \mathbf{T} \mathbf{n} \).

An equivalent referential form is deduced by multiplying (2.2.5) by \( J \) and using the identity \( J \text{div} \tilde{T} = \text{Div} \tilde{\mathbf{P}} \), where \( \tilde{\mathbf{P}} = \tilde{T} \mathbf{F}^* \) is the “effective” first Piola-Kirchhoff stress tensor. Consequently, the referential linear momentum balance equation reads

\[
\rho_\kappa \dot{\mathbf{v}} = \rho_\kappa \mathbf{b} + \text{Div} \tilde{\mathbf{P}},
\]

(2.2.7)

where

\[
\tilde{\mathbf{P}} = \mathbf{P} - \mathbf{v} \otimes \mathbf{M} + \tilde{T} \mathbf{F}^*,
\]

(2.2.8)

and \( \mathbf{P} \) is the standard first Piola-Kirchhoff stress tensor defined by \( \mathbf{P} = \mathbf{T} \mathbf{F}^* \). Computationally, it often proves convenient to use the second Piola-Kirchhoff stress \( \tilde{\mathbf{S}} \), defined by the relation

\[
\tilde{\mathbf{S}} = \mathbf{F}^{-1} \tilde{\mathbf{P}}.
\]

(2.2.9)
The balance law for angular momentum is written as

$$\frac{d}{dt} \int_P \mathbf{x} \times \rho \mathbf{v} \, dv = \int_P \mathbf{x} \times \rho \mathbf{b} \, dv + \int_{\partial P} \mathbf{x} \times \mathbf{t} \, da$$

$$+ \int_P \mathbf{x} \times (\rho \Gamma) \mathbf{v} \, dv - \int_{\partial P} \mathbf{x} \times (\mathbf{m} \cdot \mathbf{n}) \mathbf{v} \, da + \int_P \mathbf{x} \times \rho \tilde{\mathbf{b}} \, dv + \int_{\partial P} \mathbf{x} \times \tilde{\mathbf{t}} \, da.$$  \hspace{1cm} (2.2.10)

Using, again, the Reynolds’ transport formula and taking into account the local mass and linear momentum balance equations, (2.1.8) and (2.2.5), equation (2.2.10) simplifies to

$$\int_P \mathbf{x} \times \text{div} \mathbf{T} \, dv = \int_{\partial P} \mathbf{x} \times \mathbf{T} \mathbf{n} \, da.$$  \hspace{1cm} (2.2.11)

This, in turn, leads to the local form of the spatial angular momentum balance equation

$$\mathbf{T} = \mathbf{T}^T.$$  \hspace{1cm} (2.2.12)

An equivalent referential local form of angular momentum balance is expressed as

$$\overline{\mathbf{P}} \mathbf{F}^T = \mathbf{F} \overline{\mathbf{P}}^T.$$  \hspace{1cm} (2.2.13)

It can be readily concluded from (2.2.5), (2.2.6) and (2.2.12) that if there are no mass flux and irreversible force contributions (i.e., if $\mathbf{m} = \tilde{\mathbf{b}} = \tilde{\mathbf{t}} = \mathbf{0}$), then the effective body force and stress measures as well as the momentum balance statements reduce to their standard counterparts (i.e., $\overline{\mathbf{b}} \mapsto \mathbf{b}$, $\mathbf{T} \mapsto \mathbf{T}$, and $\overline{\mathbf{P}} \mapsto \mathbf{P}$). It should be noted, however, that the volumetric mass source $\Gamma$ need not be zero for this reduction to occur.

2.3 Energy balance

A statement of mechanical energy balance can be deduced in the form

$$\frac{d}{dt} \left[ \int_P \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_P \mathbf{T} \cdot \mathbf{L} \, dv \right] =$$

$$\int_P \rho \tilde{\mathbf{b}} \cdot \mathbf{v} \, dv + \int_{\partial P} \tilde{\mathbf{t}} \cdot \mathbf{v} \, da + \int_P \frac{1}{2} (\rho \Gamma - \text{div} \mathbf{m}) \mathbf{v} \cdot \mathbf{v} \, dv$$  \hspace{1cm} (2.3.1)

by taking the scalar product of the linear momentum balance (2.2.5) with the velocity, integrating over an arbitrary region $\mathcal{P} \subset \mathcal{R}$, using the mass balance statement (2.1.8), and invoking the divergence theorem. Note that the last term on the right-hand side of (2.3.1) is the supply of kinetic energy in $\mathcal{P}$ due to added mass.

A general energy balance statement is postulated in the form

$$\frac{d}{dt} \left[ \int_P \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_P \rho u \, dv \right] =$$

$$\int_P \rho \tilde{\mathbf{b}} \cdot \mathbf{v} \, dv + \int_{\partial P} \tilde{\mathbf{t}} \cdot \mathbf{v} \, da$$

$$+ \int_P \rho (r + r_i) \, dv - \int_{\partial \mathcal{P}} (\mathbf{q} + \mathbf{q}_i) \cdot \mathbf{n} \, da + \int_P \frac{1}{2} (\rho \Gamma - \text{div} \mathbf{m}) \mathbf{v} \cdot \mathbf{v} \, dv + \int_P (\rho \Gamma - \text{div} \mathbf{m}) \tilde{u} \, dv.$$  \hspace{1cm} (2.3.2)
Here, \( u \) is the internal energy per unit mass of the existing mass, \( \tilde{u} \) is the internal energy per unit mass of the added mass, \( r \) is the heat supply per unit mass, and \( q \) is the heat flux into \( P \) per unit surface area. In addition, the irreversible heating terms \( r_i \) (per unit mass) and \( q_i \) (per unit surface area) account for energy expended in the growth process itself, such as energy to adhere or bind the new mass to the existing body, frictional energy lost in mass locomotion, and energy expended in mass transport. Note that the last integral on the right-hand side of (2.3.2) is the rate at which internal energy is added to the region \( P \) with the new mass, possibly with a different energy content \( \tilde{u} \) than that of the existing mass at a material point.

Again, in connection to [18], an equivalent form of the energy statement can be derived by setting \( \tilde{u} = (\tilde{u} - u) + u \) in (2.3.2). This substitution leads to

\[
\frac{d}{dt} \left[ \int_{\mathcal{P}} \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \rho u \, dv \right] = \int_{\mathcal{P}} \rho \mathbf{B} \cdot \mathbf{v} \, dv + \int_{\partial \mathcal{P}} \mathbf{T} \cdot \mathbf{n} \, da \\
+ \int_{\mathcal{P}} \rho (r + \tilde{r}) \, dv - \int_{\partial \mathcal{P}} \left( \mathbf{q} + \tilde{q} \right) \cdot \mathbf{n} \, da + \int_{\mathcal{P}} \frac{1}{2} \left( \rho \Gamma - \text{div} \, \mathbf{m} \right) \mathbf{v} \cdot \mathbf{v} \, dv + \int_{\mathcal{P}} \left( \rho \Gamma - \text{div} \, \mathbf{m} \right) u \, dv.
\]

(2.3.3)

where

\[
\tilde{r} = (\rho \Gamma - \text{div} \, \mathbf{m})(\tilde{u} - u) + r_i, \quad \tilde{q} = q_i.
\]

(2.3.4)

In this representation, \( \tilde{r} \) is interpreted as the total irreversible heat supply, subsuming all the effects of the added mass having a different energy content. In the special case where the added mass has internal energy identical to the existing mass and there are no other dissipative effects associated with the growth process, \( \tilde{r} \mapsto 0 \) and \( \tilde{q} \mapsto 0 \). No additional irreversible terms were needed in the mechanical energy balance (2.3.1) because the mechanical irreversibilities which arise from the new mass entering the system with a different momentum are already accounted for in the “effective” terms \( \mathbf{B} \) and \( \mathbf{T} \).

Subtracting the mechanical energy balance (2.3.1) from the total energy balance (2.3.2) and applying both the Reynolds’ transport theorem and the localization theorem produces the local form of the spatial thermal energy balance equation

\[
\rho \dot{u} = \mathbf{T} \cdot \mathbf{L} + \rho (r + \tilde{r}) - \text{div} (\mathbf{q} + \tilde{q}).
\]

(2.3.5)

The equivalent referential thermal energy balance equation is

\[
\rho_\kappa \dot{u} = \mathbf{F} \cdot \dot{\mathbf{F}} + \rho_\kappa (r + \tilde{r}) - \text{Div} (\mathbf{q}_\kappa + \tilde{q}_\kappa),
\]

(2.3.6)

where, \( \mathbf{q}_\kappa = J \mathbf{F}^{-1} \mathbf{q} \), and \( \tilde{q}_\kappa = J \mathbf{F}^{-1} \tilde{q} \) are the referential heat flux vectors.
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Again, it is clear that in the absence of growth, the thermal energy balance equation reduces to the standard form.

3 Constitutive Theory

The balances of mass, linear momentum, angular momentum, and energy provide eight equations for twenty-six unknown fields \( \{ \chi, \rho, \overline{T}, u, q, \theta, \Gamma, m, \bar{r}, \bar{q} \} \). Here, it is assumed that the conventional component of the body force \( b \) and the heating source \( r \) are known. Also, \( \theta \) is the absolute temperature of the body at a material point. The first eighteen unknown fields are the conventional set for a thermoelastic material. The remaining eight unknowns, four describing mass sources and mass fluxes and four describing irreversibilities, extend the model to thermoelastic-growth materials. Constitutive laws which describe the behavior of an idealized material and relate the kinematic, mechanical, thermal, and growth fields furnish the remaining relations necessary to mathematically close the system of equations.

3.1 Multiplicative decomposition of the deformation gradient

In modeling thermoelastic-growth materials at finite deformation, a multiplicative decomposition of the deformation gradient may be considered in the form

\[
F = F_e F_g.
\]  

(3.1.1)

The two factors \( F_e \) and \( F_g \) are not, in general, the gradient of a deformation field, therefore do not satisfy compatibility conditions. The factor \( F_g \) is a local mapping of a material neighborhood in the reference configuration, \( \mathcal{R}_\kappa \), to a local maximally unloaded intermediate configuration [30]. The intermediate configuration \( \mathcal{R}_g \) is the collection of these local intermediate configurations and is generally incompatible. The intermediate configuration captures growth deformation. The factor \( F_e \) is an elastic mapping of the local material neighborhoods in the intermediate configuration into the global, compatible current configuration \( \mathcal{R} \) of the body (Figure 2).

On physical grounds, the deformation gradient decomposition (3.1.1) implies that mass is preserved from \( \mathcal{R}_g \mapsto \mathcal{R} \), namely

\[
\mathcal{M}(\mathcal{R}_g) = \mathcal{M}(\mathcal{R}),
\]  

(3.1.2)
which, upon localization, and use of (2.1.3) yields
\[ \rho_\kappa = \rho_g J_g. \]  
(3.1.3)

Here, \( \rho_g \) is the growth density, the density in the intermediate configuration, and \( J_g = \det F_g \) is the Jacobian of the growth deformation gradient. Taking the material time derivative of (3.1.3) leads to
\[ \dot{\rho}_\kappa = \dot{\rho}_g J_g = \dot{\rho}_g + \dot{\rho}_g (J_g - 1) + (\rho_g - \rho_0) \dot{J}_g + \rho_0 \dot{J}_g. \]  
(3.1.4)

Two special cases frequently considered are density-preserving growth and volume-preserving growth (densification). In density-preserving growth, it is assumed that mass growth occurs through volume change of the body at constant density, \( \rho_g = \rho_0 \). In this case, (3.1.4) reduces to
\[ \dot{\rho}_\kappa = \rho_0 \dot{J}_g. \]  
(3.1.5)

In volume-preserving growth, on the other hand, it is assumed that mass growth occurs through density change of the body at constant volume, \( J_g = 1 \). In this case, (3.1.4) reduces to
\[ \dot{\rho}_\kappa = \dot{\rho}_g. \]  
(3.1.6)

General mass growth permits simultaneous density and volume changes, as indicated by the second and third term on the right-hand side of (3.1.4) which couple the two special growth cases.

By comparing the balance of mass equation (2.1.9) with equation (3.1.4), it can be seen that, as an alternative to constitutively specifying the set \( \{ \Gamma, \mathbf{M} \} \), it is possible to consider instead the set \( \{ \rho_g, J_g \} \) as the independent constitutive variables. However, it should be noted that there does not exist a bijective relation between the two sets of variables. For example, if \( \rho_g \) and \( J_g \) are given, then equating (2.1.9) to (3.1.4) does not provide a unique prescription for \( \Gamma \) and \( \mathbf{M} \). If the division of growth into volumetric sources and surface fluxes is of interest, then specification of the mass flux vector \( \mathbf{M} \) closes the system and uniquely determines the volumetric source \( \Gamma \). This prescription also permits the total stress (2.2.8) to be separated into its individual contributions. If, instead, the volumetric source \( \Gamma \) is prescribed, the surface flux vector \( \mathbf{M} \) is (non-uniquely) specified through its divergence. Decomposition of the total stress (2.2.8) is not possible in this case.

The approach based on the set \( \{ \rho_g, J_g \} \) allows for trivial recovery of the special growth
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cases and is well-suited for computational implementation; it will be the formulation used in the ensuing developments. Depending on the completeness of the information sought about volumetric sources, mass fluxes, and stress components, either $\Gamma$ or $\mathbf{M}$ must also be constitutively specified.

3.2 Constitutive framework for thermoelastic-growth materials

For the purposes of this paper, growth is modeled as a stress-induced process; that is, the material behaves thermoelastically until an activation stress initiates growth (or resorption). Experimental results for arteries, muscle, and bone tissue indicate growth as a result of increased tissue stress levels and resorption as a result of decreased tissue stress levels [6, 31, 32].

Consider an arbitrary material point $X$ at a homeostatic Piola-Kirchhoff stress state $\mathbf{S}_0$ and absolute temperature $\theta_0$ in a body $B$ of thermoelastic-growth material in stress-temperature space $\mathcal{S} = \{\mathbf{S}, \theta\}$. A homeostatic state is a particular stable state of the body from a set of possible stable states. If subjected to small perturbations from the homeostatic state, the body naturally returns to this state through intrinsic biological regulation. The theory of finite growth is based on the following hypotheses (Figure 3):

a. The existence of an open set $\mathcal{S}_0 \subset \mathbb{R}^7$ in stress-temperature space containing $(\mathbf{S}_0, \theta_0)$, such that the material behaves in a conventional thermoelastic manner within this region. Here, $\mathcal{S}_0$ is assumed to be simply-connected and bounded by a smooth, oriented hypersurface $\partial \mathcal{S}_0$, called the homeostatic surface.

b. The existence of an open set $\mathcal{S}_r \subset \mathbb{R}^7$ in stress-temperature space containing $\mathcal{S}_0$: $\mathcal{S}_0 \subseteq \mathcal{S}_r$. As in part a., $\mathcal{S}_r$ is assumed to be simply-connected and bounded by a smooth, oriented hypersurface $\partial \mathcal{S}_r$, called the resorption activation surface.

c. The existence of an open set $\mathcal{S}_g \subset \mathbb{R}^7$ in stress-temperature space containing $\mathcal{S}_r$: $\mathcal{S}_r \subset \mathcal{S}_g$. Again, $\mathcal{S}_g$ is assumed to be simply-connected and bounded by a smooth, oriented hypersurface $\partial \mathcal{S}_g$, called the growth activation surface.

The region $\mathcal{S}_0$ models a range of normal stress-temperature activity or loading about the homeostatic state which produces no growth response. In this region, the material behaves as a conventional thermoelastic material; accordingly, it is expected that the internal energy $u$, stress $\mathbf{S}$, and heat flux $\mathbf{q}_h$ will depend only on the total Lagrangian strain $\mathbf{E}$ and
temperature \( \theta \) as described by constitutive equations of the type

\[
\mathbf{u} = \hat{\mathbf{u}}(\mathbf{E}, \theta), \quad \mathbf{S} = \hat{\mathbf{S}}(\mathbf{E}, \theta), \quad q_\kappa = \hat{q}_\kappa(\mathbf{E}, \theta, \text{Grad} \theta),
\]

(3.2.1)

where the heat flux response function is subject to the condition

\[
\hat{q}_\kappa(\mathbf{E}, \theta, 0) = \mathbf{0}.
\]

(3.2.2)

The total Lagrangian strain is defined by \( \mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) \) where \( \mathbf{I} \) is the second-order referential identity tensor.

For a thermoelastic-growth material, changes in strain and temperature may initiate growth (or resorption) at the activation surface \( \partial \mathcal{S}_g \) (or \( \partial \mathcal{S}_r \)). Growth processes alter the stress state and internal energy content of the body; the form of the constitutive equations must reflect this coupling. To characterize growth, assume that the constitutive functions depend on the growth deformation gradient, a scalar measure of isotropic growth-induced hardening \( \kappa \), a symmetric second-order tensor measure of kinematic growth-induced hardening \( \alpha \), and the growth density in addition to the total deformation gradient and temperature. The growth variables are defined on the intermediate configuration. It follows that for a general thermoelastic-growth material,

\[
\mathbf{u} = \hat{\mathbf{u}}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \quad \mathbf{S} = \hat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g), \quad q_\kappa = \hat{q}_\kappa(\mathbf{F}, \theta, \text{Grad} \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g).
\]

(3.2.3)

The constitutive responses (3.2.3) must satisfy invariance requirements. In particular, the second Piola-Kirchhoff stress must satisfy the transformation

\[
\hat{\mathbf{S}}(\mathbf{F}, \theta, \mathbf{F}_g, \kappa, \alpha, \rho_g) = \hat{\mathbf{S}}(\mathbf{F}^+, \theta^+, \mathbf{F}_g^+, \kappa^+, \alpha^+, \rho_g^+).
\]

(3.2.4)

For a rigid-body rotation \( \mathbf{Q} \) of the current configuration, it can be shown that

\[
\mathbf{F}^+ = \mathbf{QF},
\]

(3.2.5)

while the temperature is assumed to be frame-indifferent, namely

\[
\theta^+ = \theta.
\]

(3.2.6)

Using decomposition (3.1.1) with (3.2.5) implies the deformation gradient factors transform as

\[
\mathbf{F}_e^+ = \mathbf{QF}_e \mathbf{\tilde{Q}}, \quad \mathbf{F}_g^+ = \mathbf{QF}_g.
\]

(3.2.7)
where $\tilde{Q}$ is a rotation, generally distinct from $Q$ [17, 33, 34]. Assume the remaining growth variables transform according to the rules

$$
\kappa^+ = \kappa, \quad \alpha^+ = \alpha, \quad \rho_g^+ = \rho_g.
$$

(3.2.8)

Substituting (3.2.5), (3.2.6) (3.2.7), and (3.2.8) into (3.2.4) produces

$$
\tilde{S}(F, \theta, F_g, \kappa, \alpha, \rho_g) = \tilde{S}(QF, \theta, \tilde{Q}F_g, \kappa, \alpha, \rho_g).
$$

(3.2.9)

Using the unique polar decompositions $F = RU$ and $F_g = R_g U_g$, where $R$ and $R_g$ are rotations and $U$ and $U_g$ are the associated right stretch tensors, yields

$$
\tilde{S}(F, \theta, F_g, \kappa, \alpha, \rho_g) = \tilde{S}(QRU, \theta, \tilde{Q}R_g U_g, \kappa, \alpha, \rho_g).
$$

(3.2.10)

By selecting $Q = R^T$, $\tilde{Q} = R_g^T$, and using the relations $U = \hat{U}(E)$ and $U_g = \hat{U}_g(E_g)$, the response function (3.2.3) is restricted to the form

$$
\tilde{S} = \tilde{S}(E, \theta, G),
$$

(3.2.11)

where $G = (E_g, \kappa, \alpha, \rho_g)$ is the set of growth variables. Here $E_g = \frac{1}{2}(F_g^T F_g - I)$ is the growth strain. Similar invariance arguments produce the restricted form of the remaining response functions:

$$
u = \hat{u}(E, \theta, G), \quad q_\kappa = \hat{q}_\kappa(E, \theta, \text{Grad} \, \theta, G).
$$

(3.2.12)

Furthermore, assume the irreversible heat supply and heat flux response functions have the same dependencies, namely

$$
r_i = \hat{r}_i(E, \theta, G), \quad q_{\kappa i} = \hat{q}_{\kappa i}(E, \theta, \text{Grad} \, \theta, G),
$$

(3.2.13)

subject to the restrictions

$$
\hat{r}_i(E, \theta, G)|_{\dot{G} = 0} = 0, \quad \hat{q}_{\kappa i}(E, \theta, \text{Grad} \, \theta, G)|_{\dot{G} = 0} = 0.
$$

(3.2.14)

Recall that the irreversible heat supply and heat flux account for energy dissipation due to growth processes and the difference in internal energy content between the existing and added mass at a material point. The restrictions (3.2.14) are statements of the fact that if there is no growth or resorption ($\dot{G} = 0$), then no energy is expended in growth processes and no mass, with its associated internal energy, is entering or leaving the body.
Notice that for fixed growth variables, the constitutive equations (3.2.11) and (3.2.12), recover those of a conventional thermoelastic material (3.2.1). That is, a thermoelastic-growth material may be regarded as a family of thermoelastic materials parameterized by the growth variables $G$.

To complete the constitutive modeling, rate or evolution equations must be assumed for the set of growth variables $G$. When growth (resorption) occurs, let the evolution of the growth variables be of the form

$$
\dot{E}_g = \lambda \zeta, \quad \dot{\kappa} = \lambda \sigma, \quad \dot{\alpha} = \lambda \omega, \quad \dot{\rho}_g = \delta.
$$

(3.2.15)

Here $\zeta$ and $\omega$ are prescribed second-order tensor functions of the variables $(E, \theta, G)$ while $\sigma$ and $\delta$ are prescribed scalar functions of the variables $(E, \theta, G)$. Also, $\lambda$ is a scalar Lagrange multiplier enforcing constraints on the growth and resorption activation surfaces to be specified in Section 4.

Assuming $\hat{S}(E, \theta, G)$ is invertible for fixed values of $\theta$ and $G$, the Lagrangian strain may be expressed as

$$
E = \hat{E}(S, \theta, G).
$$

(3.2.16)

This relation allows for any constitutive function described in strain-temperature space to be transformed to stress-temperature space and vice versa.

### 3.3 Thermodynamics and entropy balance

Thermodynamic considerations place additional restrictions on the constitutive equations. Following the procedure proposed by Rivlin [35, 36] as employed by Casey for elastic-thermo-plastic materials at finite deformation [37], an entropy function will be constructed and the Second Law of Thermodynamics invoked to obtain restrictions. Consider first a thermoelastic-growth material undergoing an arbitrary homothermal process at fixed $G$.

The energy equation (2.3.6) for this process reduces to

$$
\rho \kappa \dot{u} = \mathbf{F} \cdot \dot{\mathbf{F}} + \rho \kappa r = \mathbf{S} \cdot \dot{E} + \rho \kappa r.
$$

(3.3.1)

A consequence of the Second Law of Thermodynamics is the path-independence of the Clausius integral [38, Section 9.10],

$$
\int_{t_0}^{t} \frac{r}{\theta} \, dt = \int_{t_0}^{t} \frac{1}{\theta} \left( \dot{u} - \frac{\mathbf{S} \cdot \dot{E}}{\rho \kappa} \right) \, dt.
$$

(3.3.2)
Path-independence suggests defining a potential \( \eta = \dot{\eta}(E, \theta, G) \), the entropy function, such that

\[
\dot{\eta} = \frac{r}{\theta} = \frac{1}{\theta} \left( \dot{u} - \frac{\mathbf{S} \cdot \mathbf{E}}{\rho_\kappa} \right),
\]

(3.3.3)

for all homothermal processes at fixed \( G \). Introducing the Helmholtz free energy

\[
\psi = \dot{\psi}(E, \theta, G) = u - \eta \theta,
\]

(3.3.4)

the Gibbs equation for a homothermal process at fixed \( G \) can be derived from (3.3.1) as

\[
\rho_\kappa \dot{\psi} = \mathbf{S} \cdot \dot{\mathbf{E}} - \rho_\kappa \eta \dot{\theta}.
\]

(3.3.5)

Expanding the material derivatives in (3.3.5) leads to

\[
\rho_\kappa \left( \frac{\partial \dot{\psi}}{\partial \theta} + \eta \right) \dot{\theta} + \left( \rho_\kappa \frac{\partial \dot{\psi}}{\partial E} \mathbf{S} \right) \cdot \dot{\mathbf{E}} = 0,
\]

(3.3.6)

which is valid for all values of \( \dot{E} \) and \( \dot{\theta} \). Varying \( \dot{E} \) and \( \dot{\theta} \) independently and noting the coefficients of \( \dot{\mathbf{E}} \) and \( \dot{\theta} \) to be rate-independent yields the Gibbs relations

\[
\eta = \dot{\eta}(E, \theta, G) = -\frac{\partial \dot{\psi}}{\partial \theta}, \quad \mathbf{S} = \mathbf{S}(E, \theta, G) = \rho_\kappa \frac{\partial \dot{\psi}}{\partial E},
\]

(3.3.7)

for homothermal processes at fixed \( G \). The Gibbs relations were derived under the restrictions of homothermal processes at fixed growth parameters \( G \); however, (3.3.7)\(_{1,2}\) do not depend on \( \text{Grad} \theta \) and remain valid for arbitrary thermoelastic processes at fixed \( G \). Additionally, (3.3.7)\(_{1,2}\) are independent of the rates of \( E_g, \kappa, \alpha \), and \( \rho_g \) from which it is concluded that the Gibbs relations remain valid even for changing \( G \). That is, the Gibbs relations extend to thermoelastic-growth materials for arbitrary processes.

For a general thermoelastic-growth process, the energy equation (2.3.6), written in terms of the Helmholtz free energy and entropy and reduced by Gibbs relations (3.3.7), is

\[
\rho_\kappa \dot{\eta} \theta = \rho_\kappa (r + \tilde{r}) - \text{Div}(\mathbf{q}_\kappa + \tilde{\mathbf{q}}_\kappa) - \rho_\kappa \left( \frac{\partial \dot{\psi}}{\partial E_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \dot{\psi}}{\partial \kappa} \dot{\kappa} + \frac{\partial \dot{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \dot{\psi}}{\partial \rho_g} \dot{\rho}_g \right).
\]

(3.3.8)

Integrating (3.3.8) over an arbitrary material region \( \mathcal{P}_\kappa \) and using Reynolds’ transport theorem results in the global balance of entropy statement

\[
\frac{d}{dt} \int_{\mathcal{P}_\kappa} \rho_\kappa \eta dV = \int_{\mathcal{P}_\kappa} \rho_\kappa \frac{r + \tilde{r}}{\theta} dV - \int_{\partial \mathcal{P}_\kappa} \frac{\mathbf{q}_\kappa + \tilde{\mathbf{q}}_\kappa}{\theta} dA - \int_{\mathcal{P}_\kappa} \theta^{-2} (\mathbf{q}_\kappa + \tilde{\mathbf{q}}_\kappa) \cdot \text{Grad} \theta dV
\]

\[
- \int_{\mathcal{P}_\kappa} \rho_\kappa \left( \frac{\partial \dot{\psi}}{\partial E_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \dot{\psi}}{\partial \kappa} \dot{\kappa} + \frac{\partial \dot{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \dot{\psi}}{\partial \rho_g} \dot{\rho}_g \right) dV + \int_{\mathcal{P}_\kappa} (\rho_\kappa \Gamma - \text{Div} \mathbf{M}) \eta dV.
\]

(3.3.9)
Assume that a modified Clausius-Duhem inequality is a valid form of the Second Law of Thermodynamics for thermoelastic-growth materials, according to

\[
\frac{d}{dt} \int_{P_\kappa} \rho_\kappa \eta \, dV \geq \int_{P_\kappa} \rho_\kappa \frac{r}{\theta} \, dV - \int_{\partial P_\kappa} \frac{q_\kappa}{\theta} \, dA + \int_{P_\kappa} (\rho_\kappa \Gamma - \text{Div} M) \eta \, dV.
\]  

(3.3.10)

This statement incorporates the additional entropy entering with new mass in the last integral. Localization of the modified Clausius-Duhem inequality produces

\[
\rho_\kappa \dot{\eta} \theta \geq \rho_\kappa r - \text{Div} q_\kappa + \theta^{-1} q_\kappa \cdot \text{Grad} \theta,
\]  

(3.3.11)

which, when applied to (3.3.8), yields the inequality

\[
\rho_\kappa \dot{r} - \text{Div} \tilde{q}_\kappa - \rho_\kappa \left( \frac{\partial \tilde{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \tilde{\psi}}{\partial \kappa} \dot{\kappa} + \frac{\partial \tilde{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \tilde{\psi}}{\partial \rho_g} \dot{\rho}_g \right) - \theta^{-1} q_\kappa \cdot \text{Grad} \theta \geq 0.
\]  

(3.3.12)

Considering an arbitrary homothermal process with no irreversible heating gives rise to

\[
\frac{\partial \tilde{\psi}}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial \tilde{\psi}}{\partial \kappa} \dot{\kappa} + \frac{\partial \tilde{\psi}}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial \tilde{\psi}}{\partial \rho_g} \dot{\rho}_g \leq 0.
\]  

(3.3.13)

Again, noting that all terms are independent of the temperature gradient leads to the conclusion that inequality (3.3.13) holds for all processes without irreversible heating. Considering a thermoelastic process with no irreversible heating produces the standard heat-conduction inequality

\[
-q_\kappa \cdot \text{Grad} \theta \geq 0.
\]  

(3.3.14)

This inequality was derived at fixed \( \mathcal{G} \); however, noting \( q_\kappa \) and \( \text{Grad} \theta \) are rate independent, inequality (3.3.14) holds for growth processes as well.

4 Growth Activation Criteria

In Section 3.2, the homeostatic surface \( \partial \mathcal{S}_0 \), the resorption activation surface \( \partial \mathcal{S}_r \), and the growth activation surface \( \partial \mathcal{S}_g \) were introduced. These surfaces were used to define a family of thermoelastic materials; each thermoelastic material in the family corresponds to a fixed set of growth variables \( \mathcal{G} \). The parametrizing growth variables (and thus the specific thermoelastic material) change during growth (resorption) processes which are initiated at the growth activation surface (resorption activation surface). Prescriptions for the homeostatic and activation surfaces, including loading directionality and penetrability criteria for each surface, are now formulated.
4.1 Homeostatic surface

Let the homeostatic hypersurface \( \partial S_0 \) in stress-temperature space \( S \) be defined by

\[
f_0(\mathbf{S}, \theta, \mathcal{G}) = 0, \quad (4.1.1)
\]

where \( f_0 \) is a differentiable, real-valued function. The homeostatic function can be transformed from stress-temperature space to strain-temperature space \( \mathcal{E} \),

\[
f_0(\mathbf{S}, \theta, \mathcal{G}) = f_0(\mathbf{S}(\mathbf{E}, \theta, \mathcal{G}), \theta, \mathcal{G}) = g_0(\mathbf{E}, \theta, \mathcal{G}), \quad (4.1.2)
\]

and, conversely, using the invertibility of the stress response,

\[
g_0(\mathbf{E}, \theta, \mathcal{G}) = g_0(\mathbf{E}(\mathbf{S}, \theta, \mathcal{G}), \theta, \mathcal{G}) = f_0(\mathbf{S}, \theta, \mathcal{G}). \quad (4.1.3)
\]

Material behavior is thermoelastic for all state variables belonging to the set

\[
S_0 = \{(\mathbf{S}, \theta, \mathcal{G}) \mid f_0(\mathbf{S}, \theta, \mathcal{G}) < 0\}. \quad (4.1.4)
\]

The utility of the surface is in defining a space for thermoelastic material behavior about a homeostatic state for normal loading behavior which stimulates no growth response. The actual state may or may not be contained in \( S_0 \); that is, a trajectory in stress-temperature space penetrating the homeostatic surface may move the state either outwards from inside the homeostatic space or inwards from outside the homeostatic space. The homeostatic surface is penetrable in both directions.

Defining a load indicator as the inner product between the normal vector to the surface in strain-temperature space and the tangent to the strain-temperature trajectory, namely

\[
\dot{g}_0 = \frac{\partial g_0}{\partial \mathbf{E}} \cdot \mathbf{E} + \frac{\partial g_0}{\partial \theta} \dot{\theta}, \quad (4.1.5)
\]

the direction of loading on the surface (\( g_0 = 0 \)) can be identified as

\[
\begin{cases} 
\dot{g}_0 > 0 & \iff \text{elastic loading from homeostatic space}, \\
\dot{g}_0 = 0 & \iff \text{neutral loading on homeostatic surface}, \\
\dot{g}_0 < 0 & \iff \text{elastic unloading into homeostatic space}.
\end{cases} \quad (4.1.6)
\]

4.2 Resorption activation surface

Let the resorption hypersurface \( \partial S_r \) in stress-temperature space be defined by

\[
f_r(\mathbf{S}, \theta, \mathcal{G}) = 0, \quad (4.2.1)
\]
where \( f_r \) is a differentiable, real-valued function subject to the restrictions

\[
\begin{align*}
  f_r & = f_0 \quad \text{for } G = \{I, 0, 0, \rho_0\}, \\
  f_r & \leq f_0 \quad \text{otherwise}.
\end{align*}
\]  

(4.2.2)

The restriction \((4.2.2)_1\) requires the resorption surface to coincide with the homeostatic surface prior to any growth. As with the homeostatic function, the resorption function can be transformed between stress-temperature space and strain-temperature space,

\[
  f_r(\bar{S}, \theta, G) = f_r(\hat{S}(E, \theta, G), \theta, G) = g_r(E, \theta, G).
\]  

(4.2.3)

The space enclosed by the resorption surface is

\[
  S_r = \{(\bar{S}, \theta, G) \mid f_r(\bar{S}, \theta, G) \leq 0\}.
\]  

(4.2.4)

The resorption surface is also penetrable in both directions under certain constraints as discussed below. Defining a load indicator,

\[
  \hat{g}_r = \frac{\partial g_r}{\partial E} \cdot \dot{E} + \frac{\partial g_r}{\partial \theta} \dot{\theta},
\]  

(4.2.5)

the direction of loading on the surface \((g_r = 0)\) can be identified as

\[
  \begin{align*}
    \hat{g}_r & > 0 \iff \text{elastic loading through resorption surface}, \\
    \hat{g}_r & = 0 \iff \text{neutral loading on resorption surface}, \\
    \hat{g}_r & < 0 \iff \text{resorption}.
  \end{align*}
\]  

(4.2.6)

A consistency condition is imposed during resorption \((g_r = 0 \text{ and } \hat{g}_r < 0)\),

\[
  \hat{g}_r = \frac{\partial g_r}{\partial E} \cdot \dot{E} + \frac{\partial g_r}{\partial \theta} \dot{\theta} + \frac{\partial g_r}{\partial \kappa} \dot{\kappa} + \frac{\partial g_r}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial g_r}{\partial \rho} \dot{\rho} = 0,
\]  

(4.2.7)

or, taking into consideration \((3.2.15)\),

\[
  \hat{g}_r + \lambda \left[ \frac{\partial g_r}{\partial \varepsilon} \cdot \dot{\varepsilon} + \frac{\partial g_r}{\partial \sigma} \cdot \dot{\sigma} + \frac{\partial g_r}{\partial \alpha} \cdot \dot{\omega} \right] + \frac{\partial g_r}{\partial \rho} = 0.
\]  

(4.2.8)

The consistency condition \((4.2.8)\) ensures, through the Lagrange multiplier \(\lambda\), that state variables evolve compatibly with the current resorption activation surface; that is, resorption from a thermoelastic-growth state \((g_r = 0)\) leads to a thermoelastic-growth state \((g_r = 0)\). This condition is enforced until the resorption surface collapses onto the homeostatic surface, \(g_r = g_0\), at which point the constraint is deactivated, allowing the admissible state variables to penetrate the resorption surface and enter the homeostatic space \(S_0\).

From the load indicator and the discussion of the consistency condition, it is observed that the resorption surface is freely penetrable by an outwards trajectory in strain-temperature space, but only penetrable by an inwards trajectory if it coincides with the homeostatic surface.
4.3 Growth activation surface

Let the growth hypersurface \( \partial S_g \) in stress-temperature space \( S \) be defined by

\[
f_g(\mathbf{S}, \theta, \mathcal{G}) = 0,
\]

where \( f_g \) is a differentiable, real-valued growth function subject to the restriction

\[
f_g < f_r.
\]

Again, the growth function can be transformed between stress-temperature space and strain-temperature space,

\[
f_g(\mathbf{S}, \theta, \mathcal{G}) = f_g(\hat{\mathbf{S}}(\mathbf{E}, \theta, \mathcal{G}), \theta, \mathcal{G}) = g_g(\mathbf{E}, \theta, \mathcal{G}).
\]

All state variables enclosed by the growth surface belong to the set

\[
S_g = \{ (\mathbf{S}, \theta, \mathcal{G}) \mid f_g(\mathbf{S}, \theta, \mathcal{G}) \leq 0 \};
\]

however, provided \( \mathcal{G} \neq \{ \mathbf{I}, \mathbf{0}, \mathbf{0}, \rho_0 \} \), admissible state variables only belong to the subset bounded by the growth and resorption surfaces

\[
S_g \setminus S_r = \{ (\mathbf{S}, \theta, \mathcal{G}) \mid f_g(\mathbf{S}, \theta, \mathcal{G}) - f_r(\mathbf{S}, \theta, \mathcal{G}) \leq 0 \}.
\]

In this subspace, material behavior is thermoelastic, growth and resorption activating at the boundaries. Defining a load indicator,

\[
\hat{g}_g = \frac{\partial g_g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial g_g}{\partial \theta} \dot{\theta},
\]

the direction of loading on the surface (\( g_g = 0 \)) can be identified as

\[
\begin{cases}
> 0 & \iff \text{growth}, \\
= 0 & \iff \text{neutral loading}, \\
< 0 & \iff \text{elastic unloading}.
\end{cases}
\]

Similar to resorption, a consistency condition is imposed during growth (\( g_g = 0 \) and \( \hat{g}_g > 0 \)) to ensure state variables evolve compatibly with the current growth activation surface,

\[
\dot{g}_g = \frac{\partial g_g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} + \frac{\partial g_g}{\partial \theta} \dot{\theta} + \frac{\partial g_g}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}}_g + \frac{\partial g_g}{\partial \mathbf{K}} \dot{\mathbf{K}} + \frac{\partial g_g}{\partial \alpha} \cdot \dot{\alpha} + \frac{\partial g_g}{\partial \rho_g} \dot{\rho}_g = 0,
\]

or with (3.2.15),

\[
\dot{g}_g + \lambda \left[ \frac{\partial g_g}{\partial \mathbf{E}_g} \cdot \dot{\mathbf{E}} + \frac{\partial g_g}{\partial \mathbf{K}} \sigma + \frac{\partial g_g}{\partial \alpha} \cdot \omega \right] + \frac{\partial g_g}{\partial \rho_g} \delta = 0.
\]

The growth surface is the only one of the three surfaces which may not be penetrated outwards by the strain-temperature state. It is also never penetrated inwards since the consistency condition (4.3.9) prohibits any state in strain-temperature space from existing outside the growth surface.
4.4 Example of activation surfaces

An application of the three-surface growth theory is schematically illustrated in Figure 4. Consider an arbitrary material point $X$ initially at a homeostatic state with stress $\bar{S}_0$ and absolute temperature $\theta_0$ undergoing a cyclic loading process which, for simplicity, is taken to be homothermal. The initial state lies within the homeostatic surface $\partial S_0$ in stress-temperature space. The resorption activation surface $\partial S_r$ is coincident with the homeostatic surface prior to growth. Upon loading, increased stress moves the state elastically outward, penetrating the homeostatic surface, until the growth activation surface $\partial S_g$ is encountered (point $(a)$). Further loading initiates growth (segment $(a)-(b)$); enforcement of the consistency condition requirement that the state remain on the active surface expands the growth activation surface. The resorption surface, coupled by the growth variables $G$, also expands. Upon unloading, decreased stress moves the state elastically inward (segment $(b)-(c)$) until the resorption activation surface is encountered (point $(c)$). Further unloading initiates resorption (segment $(c)-(d)$) and consistency condition enforcement contracts both the resorption surface and growth surface. Resorption continues along the segment $(d)-(e)$ until the homeostatic surface is encountered again at point $(e)$. At this point, the growth variables have returned to initial values, $G = \{I, 0, 0, \rho_0\}$, and the resorption and homeostatic surfaces coincide. The consistency condition is deactivated allowing the state to penetrate the common resorption and homeostatic surface and enter the homeostatic space $S_0$.

5 A Simple Model and Algorithmic Implementation of Finite Growth

A general theoretical framework for finite growth has been developed. In this section, a simple model is introduced within this framework. To focus on growth, the model will be purely mechanical such that thermodynamic variables are excluded. The model uses a principal stretch formulation with generalized measures of Lagrangian strain $E^{(m)}$ and work-conjugate stress $\bar{S}^{(m)}$; principal stretches and generalized strain and stress measures are reviewed in Appendix A.
5.1 Constitutive assumptions

For the simple model proposed, the generalized Helmholtz free energy function $\psi^{(m)}$ assumes the form

$$\rho \kappa \psi^{(m)} = \frac{1}{2} (\mathbf{E}^{(m)} - \mathbf{E}_g)^\top \mathbf{C} (\mathbf{E}^{(m)} - \mathbf{E}_g)$$

(5.1.1)

where $\mathbf{C}$ is the fourth-order material moduli tensor. The generalized stress measure computed from this Helmholtz free energy is derived in Appendix A to be

$$\bar{\mathbf{S}}^{(m)} = \mathbf{C} (\mathbf{E}^{(m)} - \mathbf{E}_g).$$

(5.1.2)

The homeostatic function, resorption function, and growth function are prescribed by

$$g_0(\mathbf{E}, \theta, \mathbf{G}) = \| \mathbf{C} (\mathbf{E}^{(m)} - \mathbf{E}_g) \| - \sigma_0$$
$$g_r(\mathbf{E}, \theta, \mathbf{G}) = \| \mathbf{C} (\mathbf{E}^{(m)} - \mathbf{E}_g) - \mathbf{\alpha} \| - (\sigma_0 + H \kappa)$$
$$g_g(\mathbf{E}, \theta, \mathbf{G}) = \| \mathbf{C} (\mathbf{E}^{(m)} - \mathbf{E}_g) - \mathbf{\alpha} \| - (\sigma_g + H \kappa),$$

(5.1.3)

respectively, where $\| \cdot \|$ denotes the tensor Euclidean norm, $\sigma_0$ the stress at the homeostatic surface, $\sigma_g (> \sigma_0)$ the initial growth activation stress, $H$ a linear isotropic material-hardening coefficient, and $\kappa$ the equivalent growth strain at time $t$ defined by

$$\kappa = \int_0^t \sqrt{\frac{2}{3}} \text{sgn}(\lambda) \| \dot{\mathbf{E}}_g \| \, d\tau.$$

(5.1.4)

The functions (5.1.3) are not only differentiable and real-valued as prescribed by the theory in Section 4, but also convex. The convexity of the functions assumes a key role in the ensuing algorithmic developments.

Evolution equations for the growth variables are required. Associative growth, the condition that the growth strain rate is parallel to the surface gradient of the active growth or resorption surface in stress-temperature space, is assumed. Associative growth implies

$$\zeta = \frac{\partial f_{(\cdot)}}{\partial \bar{\mathbf{S}}^{(m)}} = \frac{\Sigma}{\| \Sigma \|} = \mathbf{N},$$

(5.1.5)

where $f_{(\cdot)}$ is the active activation function, $f_r$ or $f_g$, and

$$\Sigma = \bar{\mathbf{S}}^{(m)} - \mathbf{\alpha}.$$

(5.1.6)

Given (3.2.15)$_1$, the rate-type constitutive equation for growth strain becomes

$$\dot{\mathbf{E}}_g = \lambda \mathbf{N}.$$
Using the fact that $\dot{E}_g = \text{sym}\{F_g^T F_g\}$, the physical interpretation of this prescription is that the symmetric part of the pull-back of the rate of growth deformation tensor to the reference configuration is proportional to the surface normal. Here, the symmetry operator for any second-order tensor $A$ is defined by $\text{sym}(A) = \frac{1}{2}[(A) + (A)^T]$.

From (5.1.4) and (5.1.7) it is concluded that

$$\dot{\kappa} = \lambda \sqrt{\frac{2}{3}},$$

(5.1.8)

and comparison with (3.2.15)$_2$ yields

$$\sigma = \sqrt{\frac{2}{3}}.$$

(5.1.9)

Kinematic hardening is also incorporated through an associative rule of the form

$$\omega = a(\kappa)N,$$

(5.1.10)

where $a(\kappa)$ is a prescribed function of the equivalent growth strain. From (3.2.15)$_3$ the rate-type constitutive equation for the kinematic hardening tensor is

$$\dot{\alpha} = \lambda a(\kappa)N.$$

(5.1.11)

The density in the intermediate configuration evolves as

$$\dot{\rho}_g = \begin{cases} 
\rho_0 \left( \frac{\|S(m)\|}{\sigma_g} \right)^{\frac{1}{k}} & \text{if } g_g = 0, \\
\rho_0 \left( \frac{\|S(m)\| + (\sigma_g - \sigma_0)}{\sigma_g} \right)^{\frac{1}{k}} & \text{if } g_r = 0, \\
0 & \text{otherwise},
\end{cases}$$

(5.1.12)

where $k$ is a material parameter. This rule is motivated by the stress state attractor stimulus model for bones of Beaupré, Orr, and Carter [39, 40]. The stress state stimulus model assumes bone mass accretion occurs when the magnitude of a characteristic stress measure exceeds that of a stimulus stress value and mass resorption occurs when the stress measure is less than the stimulus. Either of these processes results in the actual stress converging to the prescribed stress attractor state for equilibrium. Here, the rule has been modified such that the growth density is rate-independent, consistent with the modeling of the remaining growth parameters. Typical values of the exponential material parameter are $2 \leq k \leq 7$ [41].
The remaining mechanical variable to specify is the effective body force \( \mathbf{b} \) given by (2.2.6)\(_1\). Here it will be assumed that there are no mass fluxes such that \( \mathbf{m} = 0 \). With this assumption \( \mathbf{S} \mapsto \mathbf{S} \). Also, the volumetric mass source is calculated as noted in Section 3.1 by

\[
\Gamma = (\dot{\rho}_g J_g + \rho_g \dot{J}_g)/\rho_\kappa = \dot{\rho}_\kappa/\rho_\kappa = \ln(\rho_\kappa). \tag{5.1.13}
\]

The irreversible contribution \( \mathbf{\tilde{b}} \) to the effective body force \( \mathbf{b} \) is then completed, via (2.2.3)\(_1\), through the specification of \( \mathbf{\tilde{v}} \). A prescription for \( \mathbf{\tilde{v}} \), along with the traditional body force \( \mathbf{b} \), will be given for the particular applications in Section 6.

### 5.2 Algorithmic implementation

The motion of the growing body is governed by the equations of motion (2.2.5) subject to boundary conditions. Defining the displacement field \( \mathbf{u}(\mathbf{X}, t) = \chi(\mathbf{X}, t) - \mathbf{X} \), the boundary value problem can be formulated as

\[
\rho \ddot{\mathbf{u}} = \rho \mathbf{\bar{b}} + \nabla \mathbf{T} \quad \text{in } \mathcal{R}
\]

\[
\mathbf{u} = \mathbf{\bar{u}} \quad \text{on } \partial \mathcal{R}_u,
\]

\[
\mathbf{\bar{T}} \mathbf{n} = \mathbf{\bar{t}} \quad \text{on } \partial \mathcal{R}_t,
\]

where the boundary is decomposed into two disjoint subsets, \( \partial \mathcal{R}_u \) and \( \partial \mathcal{R}_t \), such that \( \partial \mathcal{R} = \partial \mathcal{R}_u \cup \partial \mathcal{R}_t \), \( \mathbf{\bar{u}} \in \mathcal{E}^3 \) is the prescribed displacement on the boundary \( \partial \mathcal{R}_u \), and \( \mathbf{\bar{t}} \in \mathcal{E}^3 \) is the prescribed traction vector on \( \partial \mathcal{R}_t \). The balance of angular momentum is satisfied \textit{a priori} by the constitutive law, therefore is not included in the system of equations.

The solution of (5.2.1) during growth requires the growth variables \( \mathcal{G} \) at time \( t \); thus, the evolution constitutive equations (3.2.15) must be integrated subject to the activation function constraint. For computational implementation, represent all second-order tensors as points in vector space \( \mathbb{R}^9 \) and define the growth vector

\[
\mathbf{Y}(t) = \begin{bmatrix} \mathbf{E}_g \\ \kappa \\ \alpha \end{bmatrix}. \tag{5.2.2}
\]

Note that the growth variable \( \rho_g \) is not included in the growth vector; \( \rho_g \) is updated subsequent to the other growth variables in a staggered integration scheme. The reason for this is the dual kinematic-constitutive nature of \( \rho_g \) as opposed to the purely kinematic response of the growth vector \( \mathbf{Y} \). Now, growth variable evolution can be formulated as the
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differential/algebraic system

\[ \dot{Y} = \lambda V(Y, t) \]

\[ g(\cdot) = 0, \quad (5.2.3) \]

subject to the initial condition

\[ Y(0) = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}. \quad (5.2.4) \]

Here, \( g(\cdot) \) is the active surface function, \( g_r \) or \( g_g \), and by comparison with (3.2.15)

\[ V(Y, t) = \begin{bmatrix} \zeta \\ \sigma \\ \omega \end{bmatrix}. \quad (5.2.5) \]

The system of differential/algebraic equations (5.2.3) will be solved using an implicit, two-step, backward-differentiation formula (BDF) integration method \([42, 43]\). Simplifying notation by indicating functional dependence on time as a subscript, \((\cdot)_n = (\cdot)(t_n)\), the BDF integration method takes the form

\[ Y_{n+1} = \beta_{1n} Y_{n-1} + \beta_{2n} Y_n + \Delta \lambda_{n+1} \gamma_n V(Y_{n+1}, t_{n+1}), \quad (5.2.6) \]

subject to

\[ g(\cdot), n+1 = 0. \quad (5.2.7) \]

Here, the coefficients \( \beta_{1n}, \beta_{2n}, \gamma_n \), are constants which specialize the method to either the first-order accurate BDF1 implicit Euler scheme \((\beta_{1n} = 0, \beta_{2n} = 1, \gamma_n = 1)\) or the second-order accurate BDF2 scheme \((\beta_{1n} = -1/3, \beta_{2n} = 4/3, \gamma_n = 2/3)\). Recall that the BDF2 values require a constant stepsize. With (5.1.5), (5.1.9), and (5.1.10) the growth vector \( Y \) may be expanded more explicitly as

\[ E_{g,n+1} = \beta_{1n} E_{g,n-1} + \beta_{2n} E_{g,n} + \gamma_n \Delta \lambda_{n+1} N_{n+1} \]

\[ \kappa_{n+1} = \beta_{1n} \kappa_{n-1} + \beta_{2n} \kappa_n + \sqrt{\frac{2}{3}} \gamma_n \Delta \lambda_{n+1} \]

\[ \alpha_{n+1} = \beta_{1n} \alpha_{n-1} + \beta_{2n} \alpha_n + \gamma_n a(\kappa_{n+1}) \Delta \lambda_{n+1} N_{n+1}, \quad (5.2.8) \]

The activation condition becomes

\[ g(\cdot) = \| C(E_{n+1}^{(m)} - E_{g,n+1}) - \alpha_{n+1} \| - (\sigma(\cdot) + H \kappa_{n+1}), \quad (5.2.9) \]

with \( \sigma(\cdot) \) taking the value \( \sigma_0 \) or \( \sigma_g \) with \( g_r \) or \( g_g \) active, respectively.
5.2.1 Elastic predictor/growth corrector on active growth surface

Consider the case where the growth surface is active, \( g_g = 0 \). Initial trial values of the growth variables are set by assuming the increment of total strain \( E^{(m)}_{n+1} - E^{(m)}_n \) is purely elastic; that is,

\[
E^{TR}_{g,n+1} = \beta_1 E_{g,n-1} + \beta_2 E_{g,n} \\
\kappa^{TR}_{n+1} = \beta_1 \kappa_{n-1} + \beta_2 \kappa_n \\
\alpha^{TR}_{n+1} = \beta_1 \alpha_{n-1} + \beta_2 \alpha_n. \tag{5.2.10}
\]

Likewise, define a trial stress

\[
\overline{S}^{(m)TR}_{n+1} = C(E^{(m)}_{n+1} - E^{TR}_{g,n+1}), \tag{5.2.11}
\]

and trial growth activation function

\[
g^{TR}_{g,n+1} = \|\overline{S}^{(m)TR}_{n+1} - \alpha^{TR}_{n+1}\| - (\sigma_g + H\kappa^{TR}_{n+1}). \tag{5.2.12}
\]

In terms of the trial values, the update equations and growth activation condition become

\[
E_{g,n+1} = E^{TR}_{g,n+1} + \gamma_n \Delta \lambda_{n+1} N_{n+1} \\
\kappa_{n+1} = \kappa^{TR}_{n+1} + \sqrt{\frac{2}{3}} \gamma_n \Delta \lambda_{n+1} \\
\alpha_{n+1} = \alpha^{TR}_{n+1} + \gamma_n a(\kappa_{n+1}) \Delta \lambda_{n+1} N_{n+1} \\
g_{g,n+1} = \|\overline{S}^{(m)TR}_{n+1} - \gamma_n \Delta \lambda_{n+1} \overline{C} N_{n+1} - \alpha^{TR}_{n+1} - \gamma_n a(\kappa_{n+1}) \Delta \lambda_{n+1} N_{n+1}\| \\
- \left[\sigma_g + H \left(\kappa^{TR}_{n+1} + \sqrt{\frac{2}{3}} \gamma_n \Delta \lambda_{n+1}\right)\right],
\]

and the stress update takes the form

\[
\overline{S}^{(m)}_{n+1} = \overline{S}^{(m)TR}_{n+1} - \gamma_n \Delta \lambda_{n+1} \overline{C} N_{n+1}. \tag{5.2.14}
\]

If the total strain increment is indeed purely elastic, the trial state is an admissible state; that is, \( g^{TR}_{g,n+1} \leq 0 \) and the consistency parameter is inactive, \( \Delta \lambda_{n+1} = 0 \). The actual growth variables, growth activation condition, and stress are equivalent to the trial values. If the total strain increment is not purely elastic, the trial state is an inadmissible state; that is, the trial growth activation function has been violated, \( g^{TR}_{g,n+1} > 0 \). The activation function violation is corrected by enforcement of the consistency condition \( \Delta \lambda_{n+1} \neq 0 \). The growth variables are updated and the activation constraint enforced
by solving the system of fourteen equations (5.2.13) in fourteen unknowns \((E_{g,n+1}, \kappa_{n+1}, \alpha_{n+1}, \Delta \lambda_{n+1})\) iteratively by Newton’s method. The actual stress is then updated using (5.2.14). This process is analogous to the classic elastic-plastic operator split return-map in rate-independent plasticity theory [44, 45]. A schematic representation of the elastic predictor/growth corrector return-mapping is illustrated in Figure 5; an elastic trial state is assumed and a growth corrector step returns the stress state to the admissible surface through a closest-point projection onto the convex activation function.

### 5.2.2 Resorption predictor/elastic corrector on active resorption surface

Consider the case where the resorption surface is active, \(g_r = 0\). Conceptually, the treatment of resorption parallels that of growth with one major difference. Resorption occurs on an unloading trajectory from the admissible state space outside the convex resorption surface, whereas growth occurs on a loading trajectory from the admissible state space inside the convex growth surface. In the case of growth, an elastic trial state violating the growth surface was corrected by projection onto the convex growth surface. If an elastic trial state was implemented for resorption, a trial state violating the resorption surface would need to be corrected by projection onto the non-convex face of the resorption surface. Numerically, this scheme is unstable due to the non-uniqueness of the projection. To ensure return-map projections onto a convex surface, the predictor/corrector method is reversed in resorption, with a resorption predictor followed by an elastic corrector. Initial trial values of the growth variables are set by assuming the increment of total strain \(E^{(m)}_{n+1} - E^{(m)}_n\) is purely resorptive,

\[
E^{(m)}_{n+1} - E^{(m)}_n = E^{TR}_{g,n+1} - E_{g,n}.
\]  

(5.2.15)

The trial growth strain is then

\[
E^{TR}_{g,n+1} = E_{g,n} + (E^{(m)}_{n+1} - E^{(m)}_n).
\]  

(5.2.16)

The trial stress (5.2.11) becomes

\[
S^{(m)TR}_{n+1} = C(E^{(m)}_n - E_{g,n}) = S^{(m)}_n ,
\]  

(5.2.17)

while the trial update equations take the form

\[
E^{TR}_{g,n+1} = \beta_1 n E_{g,n-1} + \beta_2 n E_{g,n} + \gamma_n \Delta \lambda^{TR}_{n+1} \mathbf{N}^{TR}_{n+1},
\]

\[
\kappa^{TR}_{n+1} = \beta_1 n \kappa_{n-1} + \beta_2 n \kappa_{n} + \sqrt{\frac{2}{3}} \gamma_n \Delta \lambda^{TR}_{n+1} \]

(5.2.18)

\[
\alpha^{TR}_{n+1} = \beta_1 n \alpha_{n-1} + \beta_2 n \alpha_{n} + \gamma_n a(\kappa^{TR}_{n+1}) \Delta \lambda^{TR}_{n+1} \mathbf{N}^{TR}_{n+1}.
\]
Taking the norm of (5.2.18)_1,
\[
\Delta \lambda^{TR}_{n+1} = -\frac{1}{\gamma_n} \left( E^{TR}_{g,n+1} - \beta_1 n E_{g,n-1} - \beta_2 n E_{g,n} \right),
\]
which resubstituted yields
\[
N^{TR}_{n+1} = -\frac{1}{\gamma_n} \left( E^{TR}_{g,n+1} - \beta_1 n E_{g,n-1} - \beta_2 n E_{g,n} \right),
\]
Trial values for \( \kappa^{TR}_{n+1} \) and \( \alpha^{TR}_{n+1} \) are now calculated by (5.2.18)_2,3. The trial resorption activation function is
\[
g^{TR}_{r,n+1} = \| S^{(m)TR}_{n+1} - \alpha^{TR}_{n+1} - (\sigma_0 + H \kappa^{TR}_{n+1}) \|.
\]
In terms of the trial values, the update equations and resorption activation condition become
\[
E_{g,n+1} = E^{TR}_{g,n+1} + \gamma_n (\Delta \lambda_{n+1} N_{n+1} - \Delta \lambda^{TR}_{n+1} N^{TR}_{n+1})
\]
\[
\kappa_{n+1} = \kappa^{TR}_{n+1} + \sqrt{\frac{2}{3}} \gamma_n (\Delta \lambda_{n+1} - \Delta \lambda^{TR}_{n+1})
\]
\[
\alpha_{n+1} = \alpha^{TR}_{n+1} + \gamma_n (a(\kappa^{TR}_{n+1}) \Delta \lambda_{n+1} N_{n+1} - a(\kappa^{TR}_{n+1}) \Delta \lambda^{TR}_{n+1} N^{TR}_{n+1})
\]
\[
g^{TR}_{r,n+1} = \| S^{(m)TR}_{n+1} - \gamma_n \Delta \lambda_{n+1} C N_{n+1} - \alpha^{TR}_{n+1} - \gamma_n a(\kappa^{TR}_{n+1}) \Delta \lambda_{n+1} N_{n+1}
\]
\[
+ \gamma_n \Delta \lambda^{TR}_{n+1} (C N^{TR}_{n+1} + a(\kappa^{TR}_{n+1}) N^{TR}_{n+1}) \|,
\]
and the stress update takes the form
\[
S^{(m)}_{n+1} = S^{(m)TR}_{n+1} - \gamma_n (\Delta \lambda_{n+1} C N_{n+1} - \Delta \lambda^{TR}_{n+1} C N^{TR}_{n+1}).
\]

The trial state reduces the space \( S_r \) enclosed by the resorption surface, but leaves the stress state (5.2.17) unaltered. Thus, while the trial state is always admissible, the consistency condition is violated. The violation is corrected by enforcing the consistency condition (solving (5.2.22) for \( E_{g,n+1}, \kappa_{n+1}, \alpha_{n+1}, \Delta \lambda_{n+1} \)). A schematic representation of the resorption predictor/elastic corrector return-mapping is illustrated in Figure 6; a resorptive trial state is assumed and an elastic corrector step returns the stress state to the admissible surface through a closest-point projection onto the convex activation function. The difference between the growth and resorption method can be summarized as follows: for growth, the trial state may violate the activation function and be corrected by enforcement.
of the consistency condition whereas, for resorption, the trial state \textit{a priori} violates the consistency condition and must be corrected.

For both growth and resorption, the Newton method update of growth variables is a local, internal calculation in the stress update. Once obtained, the updated stress from (5.2.14) or (5.2.23) is used to solve the equations of motion (5.2.1). The equations of motion are solved using the conventional finite element formulation; a derivation of the algorithmic tangent modulus consistent with Newton’s method for the linearized equations is provided for the proposed model in Appendix B.

6 Numerical Simulations

A simple model for finite growth was proposed in the previous section and a corresponding algorithmic model developed. The model was implemented using the nonlinear finite element program FEAP, which is partially documented in [46, 47, 48]. Selected numerical simulations are presented here: first, a uniaxial strain cycle to demonstrate the general features of the model, then representative applications to both soft and hard biomaterials. All simulations were performed with 8-node isoparametric brick elements using the implicit Euler scheme.

6.1 Homogeneous strain cycle

To illustrate qualitative aspects of the stress response and growth response of the proposed model, results for uniaxial homogeneous strain cycling are presented in Figures 7-9. For simplicity, the tests were performed in the form of a reversible, homothermal cycle with mass sources but no mass fluxes or body forces. The material was assumed isotropic and a Saint-Venant Kirchhoff model used for the elasticity tensor, \( C = \lambda_L (I \otimes I) + 2 \mu_L I \), where \( \{\lambda_L, \mu_L\} \) are the Lamé constants and \( I \) is the symmetric, fourth-order referential identity tensor. The elastic material properties in the analyses are

\[
\lambda_L = 17.3 \text{ MPa}, \quad \mu_L = 11.5 \text{ MPa}, \quad \rho_0 = 1.0 \text{ kg/m}^3, \quad m = 2.0, \quad (6.1.1)
\]

while the growth parameters, normalized by Young’s modulus, \( E = \mu_L(3\lambda_L + 2\mu_L)/(\lambda_L + \mu_L) \), are

\[
\frac{\sigma_0}{E} = 0.001, \quad \frac{\sigma_g}{E} = 0.003, \quad \frac{H}{E} = 0.10, \quad a(\kappa) = 0.0, \quad k = 2.0. \quad (6.1.2)
\]

Figure 7 depicts density-preserving material growth, \( \rho_g = \rho_0 \). The growth response is purely kinematic and depends only on the growth strain through \( J_g \). The plot consists of
a curve of the second Piola-Kirchhoff stress in the direction of straining versus elongation and a curve of the total body mass. The stress, initially zero, responds elastically within the homeostatic surface upon elongation until the growth activation stress \( \sigma_g \) is reached. At this point, growth is activated and the material response hardens. Growth persists until the strain cycle reaches maximum elongation. Upon release, the stress elastically unloads between the growth and resorption activation surfaces. At the resorption surface, the material response softens until the resorption surface collapses onto the homeostatic surface and the stress decreases elastically to zero. Corresponding to the stress response, the total body mass is initially constant during elastic elongation, increases during growth, remains constant during elastic release, and decreases during resorption.

Figure 8 depicts volume-preserving material growth, \( \dot{J}_g = 0 \). Here, the growth response is purely constitutive and depends only on the density \( \rho_g \) in the intermediate configuration via (5.1.12). The growth kinematic variables are invariant; the stress response exhibits standard hyperelastic Saint-Venant Kirchhoff behavior. Note, the particular constitutive model utilized does not distinguish between compressive and tensile stress. This feature would be necessary for certain classes of materials, such as bones, which densify under compressive loading.

Figure 9 depicts coupled density- and volume-preserving material growth; all growth parameters \( G \) are active. Mass response to compressive stress can be accretive or resorptive depending on the relative kinematic and constitutive contributions, respectively. The kinematic and constitutive contributions are controlled by the isotropic material hardening coefficient \( H \).

### 6.2 Soft tissue application: artery growth

For an application of the growth model to soft tissue biomaterials, results for a hypertensive artery are presented. A standard artery experiment induces hypertension, resulting in wall thickening and residual stress accumulation [6]. The residual stress is observed by removing a segment of artery, cutting the artery along an axial plane, and measuring the opening angle of the resulting configuration. To simulate this experiment, an artery was modeled as a uniform cylinder. Boundary conditions were applied to restrain the outer radius and ends as if the cylinder were perfectly bonded within a rigid, capped, sleeve. Hypertension was generated by increasing internal radial pressure. Again, for simplicity, the test was performed as a reversible, homothermal cycle with mass sources but no mass fluxes or body forces and an isotropic, Saint-Venant Kirchhoff material model. The elastic material
properties in the analysis are

$$\lambda_L = 6.2069 \text{ MPa}, \quad \mu_L = 0.6897 \text{ MPa}, \quad \rho_0 = 1.0 \text{ kg/m}^3, \quad m = 1.5, \quad (6.2.1)$$

while the growth parameters, normalized by Young’s modulus, are

$$\frac{\sigma_0}{E} = 0.0001, \quad \frac{\sigma_g}{E} = 0.0026, \quad \frac{H}{E} = 0.125, \quad a(\kappa) = 0.0, \quad k = 5.0. \quad (6.2.2)$$

The internal pressure was increased to produce 8.0% radial displacement, at which point, the artery was removed from the sleeve, cut along an axial plane, and permitted to open to equilibrate residual stresses. After cutting, the artery is no longer in equilibrium (assuming the cut faces are traction free) and attempts to open instantaneously.

Initially, the artery mesh consists of a rectangular sheet of elements which is formed into a cylinder (Figure 10). Along the axial cut plane, adjoining elements are constrained to have identical nodal displacements. Internal pressure is increased to produce the specified radial displacement. At this point, the artery is to be removed from the sleeve and cut along the axial plane. The large opening deformation and nonlinear response prevents convergence in the quasi-static analysis. A special procedure is devised to circumvent this difficulty. Computationally, the constraint on axial cut plane nodes is removed and the resulting reaction forces on the free surfaces stored. The reaction forces are converted to applied tractions and slowly relaxed to zero; the artery opens slowly and is numerically stable.

Figure 11 depicts the relative density change, \( \frac{\rho - \rho_0}{\rho_0} \), and final configuration of the sliced artery. Qualitatively, the simulation matches experimental results: the inner arterial wall grows the most mass and the growth process produces a residual stress field. The opening angle, as defined by Fung [49], is calculated to be 3.4°. This value accounts only for the additional residual stress generated by growth, not the homeostatic residual stress field inherent in arteries so is less than opening angles recorded in the literature.

### 6.3 Hard tissue application: bone adaptation

For an application of the growth model to hard tissue biomaterials, results for adaptation of the proximal femur are presented. As one of the most studied skeletal regions due to hip arthroplasties, there exists a large body of clinical data and models for the proximal femur; the proximal femur is an ideal candidate for evaluating bone growth. One criterion for a bone growth model is its predictive capacity to generate a normal bone density distribution from an initially homogeneous density distribution when subjected to a typical load history.
This captures the spirit of Wolff’s Law; trabecular bone patterns functionally adapt to best handle habitual loading conditions. A quantitative computed tomography (QCT, 0.2 mm in-plane and 1 mm out-of-plane voxel size) finite element model of the proximal femur is used. The elastic material properties in the analysis are

\[
\lambda_L = 555.56 \text{ MPa}, \quad \mu_L = 833.33 \text{ MPa}, \quad \rho_0 = 0.56 \text{ g/cm}^3, \quad m = 2.0, \quad (6.3.1)
\]

while the growth parameters, normalized by Young’s modulus, are

\[
\frac{\sigma_0}{E} = 0.00001, \quad \frac{\sigma_g}{E} = 0.0001, \quad \frac{H}{E} = 0.15, \quad a(\kappa) = 0.0, \quad k = 6.0. \quad (6.3.2)
\]

The density value is based on experimental measurements in the femoral neck and the Lamé constants follow from a power-law regression [50]. There is no mass flux or irreversibilities.

To model stance-type loading, the distal end was fixed, the nodes on the superior aspect of the femoral head subjected to an applied compressive force of 3000 N oriented at 20° relative to vertical [51], and the nodes on the hip abductor subjected to an applied tensile force of 1000 N oriented at 25° relative to vertical. Body forces are neglected. Figure 12 depicts the equivalent von Mises stress distribution and Figure 13 the relative density change; Figure 14 and Figure 15 are the corresponding results on an interior plane.

In this case, the relative density change \( \frac{\rho_\kappa - \bar{\rho}_\kappa}{\bar{\rho}_\kappa} \) is defined with respect to the averaged referential density \( \bar{\rho}_\kappa \) rather than the initial density \( \rho_0 \). This relative measure represents a translational shift of the growth density distribution data with respect to the arbitrarily selected initial homogeneous state. It permits the results obtained here, in which the model shifts the stress-attractor state such as to correspond with an equilibrated system for an initially unloaded configuration, to be directly compared with literature results in which the stress-attractor state corresponds with an unequilibrated system for an initially unloaded configuration [40, 27]. The translation shift normalizes the arbitrariness of the stress-attractor and homogeneous density initial configurations between the models.

Qualitatively, the simulation matches expected bone morphology and representative literature results for two-dimensional models [40, 27]. Along the primary loading path from the femoral head, a high stress trajectory forms and a corresponding strut of high density bone grows. However, it is noted that this strut is not as fully developed across the femoral head itself as in the two-dimensional simulations or bone specimens. A secondary strut of high density forms below the hip abductor. Between these two high density bone regions exists an area of low density bone achieved through resorption; these observations are in accord with anatomical data.
7 Closure

The article presents a general constitutive and computational framework for the study of rate-independent, stress-induced material growth in finitely deforming biomaterials. The equations governing the mechanics of solids capable of growth or resorption are derived. A proposed extension of classical thermodynamic theory provides a foundation for developing general constitutive relations. A multiplicative decomposition of the deformation gradient permits growth processes to be separated into a kinematic contribution and a constitutive contribution; this identification allows for the commonly utilized special cases of density-preserving growth and volume-preserving growth to be easily recovered. A three-surface activation theory is proposed to model stress-induced growth; the three surfaces effect the fundamental biological concepts of homeostatic equilibrium, resorption, and growth, respectively. Computationally, a set of growth parameters and associated evolution equations describing the three-surface model translate into a differential/algebraic system to be solved with the equations of motion. An implicit, incremental finite element formulation is developed with distinct algorithmic schemes for growth and resorption to address stability. Numerical simulations appear to perform well in generating qualitative results documented in the literature for soft and hard biomechanics applications.

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References


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Appendices

Appendix A. Generalized measures of strain and work-conjugate stress

The right Cauchy-Green deformation tensor \( C = F^T F \) has the spectral representation,

\[
C = \sum_{A=1}^{3} \lambda_A^2 (N_A \otimes N_A), \tag{A.3}
\]

where \( \lambda_A, A = 1, 2, 3 \) are the principal stretches or eigenvalues of \( U = C^{1/2} \) and \( N_A \) are the corresponding normalized eigenvectors. Using this representation, introduce generalized Lagrangian strain tensors, first considered in [52, 53], by

\[
E^{(m)} = \begin{cases} 
\frac{1}{m}(C^{m/2} - I) & \text{if } m \neq 0 \\
\frac{1}{2}\log C & \text{if } m = 0
\end{cases}, \tag{A.4}
\]

where \( C^{m/2} \) is defined for \( m \neq 0 \) by

\[
C^{m/2} = \sum_{A=1}^{3} \lambda_A^m (N_A \otimes N_A), \tag{A.5}
\]

and \( \log C \) is given by

\[
\log C = \sum_{A=1}^{3} \log \lambda_A^2 (N_A \otimes N_A). \tag{A.6}
\]

The material time derivative of the generalized strain tensor \( E^{(m)} \) is

\[
\dot{E}^{(m)} = \Xi^{(m)} \dot{E} \tag{A.7}
\]

where

\[
\Xi^{(m)} = \sum_{A=1}^{3} \lambda_A^{m-2} N_{AAAA} + \frac{1}{2} \sum_{A,B=1 \atop A \neq B}^{3} \xi(\lambda_A, \lambda_B; m) (N_{ABAB} + N_{ABBA}), \tag{A.8}
\]
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with

\[ \mathcal{N}_{ABCD} = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C \otimes \mathbf{N}_D \]  \hspace{1cm} (A.9)

and

\[ \xi(\lambda_A, \lambda_B; m) = \begin{cases} 
\frac{2(\lambda_B^m - \lambda_A^m)}{m(\lambda_B^m - \lambda_A^m)} & \text{if } \lambda_A \neq \lambda_B, m \neq 0 \\
\frac{\log(\lambda_B^m/\lambda_A^m)}{\lambda_B^m - \lambda_A^m} & \text{if } \lambda_A \neq \lambda_B, m = 0 \\
\lambda_A^{m-2} & \text{if } \lambda_A = \lambda_B
\end{cases} \]  \hspace{1cm} (A.10)

Note that \( \mathcal{E}^{(m)} \) has the minor symmetries, shown in component form as \( \mathcal{E}^{(m)}_{ABCD} = \mathcal{E}^{(m)}_{ABDC} = \mathcal{E}^{(m)}_{BACD} \), induced by the symmetry of \( \mathcal{E}^{(m)} \) and \( \mathcal{E} \). Equations (A.7)-(A.10) may be obtained by the following procedure. Start with the Green-Lagrange strain tensor in principal stretches, obtained by substituting (A.3) in (A.4) with \( m = 2 \), that is

\[ \mathbf{E} = \frac{1}{2} \left( \sum_{A=1}^{3} \lambda_A^2 \mathbf{N}_A \otimes \mathbf{N}_A - \mathbf{I} \right). \]  \hspace{1cm} (A.11)

Taking the time derivative of \( \mathbf{E} \) gives

\[ \dot{\mathbf{E}} = \frac{1}{2} \sum_{A=1}^{3} \left[ 2\lambda_A \dot{\lambda}_A \mathbf{N}_A \otimes \mathbf{N}_A + \lambda_A^2 \dot{\mathbf{N}}_A \otimes \mathbf{N}_A + \lambda_A^2 \mathbf{N}_A \otimes \dot{\mathbf{N}}_A \right]. \]  \hspace{1cm} (A.12)

Symmetry of \( \mathbf{C} \) implies the eigenvectors \( \mathbf{N}_A \) are orthogonal. Let \( \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \) be a fixed orthonormal basis in \( \mathbb{R}^3 \). Then, the eigenvectors are given by

\[ \mathbf{N}_A = \mathbf{Qe}_A, \]  \hspace{1cm} (A.13)

where \( \mathbf{Q} \) is a rotation tensor defined by the properties

\[ \mathbf{QQ}^T = \mathbf{I}, \quad \det \mathbf{Q} = 1. \]  \hspace{1cm} (A.14)

Now, the time derivative of the eigenvectors is

\[ \dot{\mathbf{N}}_A = \dot{\mathbf{Q}} \mathbf{e}_A = \dot{\mathbf{Q}} \mathbf{Q}^T \mathbf{N}_A = \Omega \mathbf{N}_A, \]  \hspace{1cm} (A.15)

where, \( \Omega = \dot{\mathbf{Q}} \mathbf{Q}^T \). From (A.14)_{1},

\[ (\mathbf{QQ}^T)^T = \dot{\mathbf{Q}} \mathbf{Q}^T + \mathbf{QQ}^T = \dot{\mathbf{Q}} \mathbf{Q}^T + (\dot{\mathbf{Q}} \mathbf{Q}^T)^T = \Omega + \Omega^T = \mathbf{0}, \]  \hspace{1cm} (A.16)

so \( \Omega^T = -\Omega \), hence \( \Omega \) is a skew-symmetric tensor. On the eigenvector basis, it has the representation \( \Omega = \Omega_{AB} \mathbf{N}_A \otimes \mathbf{N}_B \). This representation yields

\[ \dot{\mathbf{N}}_A = \Omega_{BA} \mathbf{N}_B, \]  \hspace{1cm} (A.17)
and, in turn, (A.12) becomes
\[ \dot{\mathbf{E}} = \frac{1}{2} \sum_{A=1}^{3} \left[ 2\lambda_{(A)} \dot{\lambda}_{(A)} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} + \lambda_{(A)}^2 \Omega_{BA} \mathbf{N}_{(B)} \otimes \mathbf{N}_{(A)} + \lambda_{(A)}^2 \Omega_{BA} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)} \right]. \] (A.18)

This may be rewritten as
\[ \dot{\mathbf{E}} = \sum_{A=1}^{3} \lambda_{(A)} \dot{\lambda}_{(A)} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} + \frac{1}{2} \sum_{A,B=1}^{3} \left( \lambda_{(B)}^2 - \lambda_{(A)}^2 \right) \Omega_{AB} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)}, \] (A.19)

where it was noted
\[ \sum_{A=1}^{3} \lambda_{(A)}^2 \Omega_{BA} \mathbf{N}_{(B)} \otimes \mathbf{N}_{(A)} = \sum_{B=1}^{3} \lambda_{(B)}^2 \Omega_{AB} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)}. \] (A.20)

Similarly, the time derivative of the generalized Lagrangian strain tensor for the case \( m \neq 0 \) is
\[ \dot{\mathbf{E}}^{(m)} = \sum_{A=1}^{3} \lambda_{(A)}^{m-1} \dot{\lambda}_{(A)} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} + \frac{1}{m} \sum_{A,B=1}^{3} \left( \lambda_{(B)}^m - \lambda_{(A)}^m \right) \Omega_{AB} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)}. \] (A.21)

Assume the eigenvalues \( \{\lambda_{(A)}\} \) are distinct. The components of \( \dot{\mathbf{E}} \) from (A.19) yield
\[ \dot{E}_{AA} = \lambda_{(A)} \dot{\lambda}_{(A)}, \text{ (no sum)}; \quad \Omega_{AB} = \frac{2\dot{E}_{AB}}{\lambda_{(B)}^m - \lambda_{(A)}^m} = \frac{\dot{E}_{AB} + \dot{E}_{BA}}{\lambda_{(B)}^m - \lambda_{(A)}^m}, \quad (A \neq B). \] (A.22)

Substituting (A.22) into (A.21) results in
\[ \dot{\mathbf{E}}^{(m)} = \sum_{A=1}^{3} \lambda_{(A)}^{m-2} \dot{E}_{AA} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} + \frac{1}{m} \sum_{A,B=1}^{3} \left( \lambda_{(B)}^m - \lambda_{(A)}^m \right) (\dot{E}_{AB} + \dot{E}_{BA}) \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)}, \] (A.23)

which, with the component definitions \( \dot{E}_{AB} = (\mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)}) \cdot \dot{\mathbf{E}} \), yields
\[ \dot{\mathbf{E}}^{(m)} = \left[ \sum_{A=1}^{3} \lambda_{(A)}^{m-2} \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} \otimes \mathbf{N}_{(A)} \right] + \frac{1}{m} \sum_{A,B=1}^{3} \left( \lambda_{(B)}^m - \lambda_{(A)}^m \right) \left( \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)} \otimes \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)} + \mathbf{N}_{(A)} \otimes \mathbf{N}_{(B)} \otimes \mathbf{N}_{(B)} \otimes \mathbf{N}_{(A)} \right) \dot{\mathbf{E}}. \] (A.24)

By comparison of (A.24) with (A.7), the relations (A.8) and (A.10) follow. For the case of \( m = 0 \), relation (A.10) is readily derived by the same procedure. For the case of non-distinct eigenvalues, application of L’Hôpital’s rule reduces both (A.10) to (A.10).
The Helmholtz free energy is independent of the particular strain measure selected such that
\[ \psi = \hat{\psi}(E, \theta, G) = \hat{\psi}^{(m)}(E^{(m)}, \theta, G) = \psi^{(m)}. \]  
(A.25)
The representation \( \psi^{(m)} \) of the Helmholtz free energy function can be used to construct a stress measure \( \bar{S}^{(m)} \) that is work-conjugate to \( E^{(m)} \), in the sense of Hill [54], as
\[ \bar{S}^{(m)} = \rho \frac{\partial \hat{\psi}^{(m)}}{\partial E^{(m)}}. \]  
(A.26)
Fixing growth parameters \( G \) and using the symmetry of \( \Xi^{(m)} \) with (3.3.7)\( _2 \), (A.7), and (A.26) shows that
\[ \rho \dot{\psi} = \bar{S} \cdot \dot{E} = \bar{S}^{(m)} \cdot \dot{E}^{(m)} \]
\[ = \bar{S}^{(m)} \cdot (\Xi^{(m)} \dot{E}) \]
\[ = (\Xi^{(m)} \bar{S}^{(m)}) \cdot \dot{E}, \]  
(A.27)
which implies
\[ \bar{S} = \Xi^{(m)} \bar{S}^{(m)}. \]  
(A.28)
The second Piola-Kirchhoff stress tensor is related to the generalized stress tensor by the transformation tensor \( \Xi^{(m)} \). Using (A.8), the second Piola-Kirchhoff stress tensor can be written more explicitly as
\[ \bar{S} = \sum_{A=1}^{3} \lambda^{m-2}(A) \left( N_{(A)} \cdot \bar{S}^{(m)} N_{(A)} \right) N_{(A)} \otimes N_{(A)} \]
\[ + \sum_{A,B=1}^{3} \xi(\lambda_{(A)}, \lambda_{(B)}; m) \left( N_{(A)} \cdot \bar{S}^{(m)} N_{(B)} \right) N_{(A)} \otimes N_{(B)}. \]  
(A.29)

Appendix B. Consistent algorithmic tangent modulus
Newton’s method is used to solve the discrete form of linear momentum balance. A closed-form expression for the algorithmic tangent modulus consistent with Newton’s method is derived. The referential algorithmic tangent modulus is obtained by taking the derivative of (A.28) at \( t_{n+1} \) with respect to \( E_{n+1} \),
\[ \frac{\partial \bar{S}_{n+1}}{\partial E_{n+1}} = \frac{\partial \Xi^{(m)}_{n+1}}{\partial E_{n+1}} \bar{S}^{(m)}_{n+1} + \Xi^{(m)}_{n+1} \frac{\partial \bar{S}^{(m)}_{n+1}}{\partial E_{n+1}} \Xi^{(m)}_{n+1}. \]  
(B.1)
From (A.7), it is implied that
\[
\mathbf{\Xi}_{n+1}^{(m)} = \frac{\partial \mathbf{E}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}},
\]
(B.2)
such that
\[
\mathbf{\dot{\Xi}}_{n+1}^{(m)} = \frac{\partial^2 \mathbf{E}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1} \partial \mathbf{E}_{n+1}} \mathbf{\dot{E}}_{n+1} = \frac{\partial \mathbf{\Xi}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}} \mathbf{\dot{E}}_{n+1}.
\]
(B.3)
Assume the eigenvalues \( \{\lambda_{(A)}\} \) are distinct and \( m \neq 0 \). Then, taking the time derivative of (A.8) with (A.10) and substituting (A.17), (A.22), and (A.9) yields after lengthy, but straightforward algebraic manipulations,
\[
\mathbf{\dot{\Xi}}_{n+1}^{(m)} = \sum_{A=1}^{3} (m - 2)\lambda^{m-4}_{(A)} \dot{E}_{AA} \mathbf{N}_{AAAA}
\]
\[ - \sum_{A,B=1 \atop A\neq B}^{3} \frac{\lambda^{m-2}_{(A)}}{\lambda^{2}_{(B)} - \lambda^{2}_{(A)}} (\dot{E}_{AB} + \dot{E}_{BA}) (\mathbf{N}_{AAAB} + \mathbf{N}_{AABA} + \mathbf{N}_{ABAA} + \mathbf{N}_{BAAA})
\]
\[ + \sum_{A,B=1 \atop A\neq B}^{3} \left[ \frac{\lambda^{m-2}_{(B)}}{\lambda^{2}_{(B)} - \lambda^{2}_{(A)}} - \frac{2}{m} \frac{\lambda^{m}_{(B)} - \lambda^{m}_{(A)}}{\lambda^{2}_{(B)} - \lambda^{2}_{(A)}} \dot{E}_{BB} \right] (\mathbf{N}_{ABAB} + \mathbf{N}_{ABBA})
\]
\[ + \sum_{A,B=1 \atop A\neq B}^{3} \left[ \frac{\lambda^{m}_{(A)}}{\lambda^{2}_{(B)} - \lambda^{2}_{(A)}} - \frac{2}{m} \frac{\lambda^{m}_{(A)} - \lambda^{m}_{(B)}}{\lambda^{2}_{(B)} - \lambda^{2}_{(A)}} \dot{E}_{AA} \right] (\mathbf{N}_{ABAB} + \mathbf{N}_{ABBA})
\]
\[ + \sum_{A,B,C=1 \atop A\neq B, A\neq C}^{3} \frac{1}{m} \left( \frac{\lambda^{m}_{(B)} - \lambda^{m}_{(A)}}{(\lambda^{2}_{(B)} - \lambda^{2}_{(A)})^{2}} \right) \left( \frac{\dot{E}_{CA} + \dot{E}_{AC}}{(\lambda^{2}_{(A)} - \lambda^{2}_{(C)})^{2}} \right) (\mathbf{N}_{ABBC} + \mathbf{N}_{ABCB} + \mathbf{N}_{CBBA} + \mathbf{N}_{CBAB})
\]
\[ + \sum_{A,B,C=1 \atop A\neq B, B\neq C}^{3} \frac{1}{m} \left( \frac{\lambda^{m}_{(B)} - \lambda^{m}_{(A)}}{(\lambda^{2}_{(B)} - \lambda^{2}_{(A)})^{2}} \right) \left( \frac{\dot{E}_{CB} + \dot{E}_{BC}}{(\lambda^{2}_{(B)} - \lambda^{2}_{(C)})^{2}} \right) (\mathbf{N}_{ABAC} + \mathbf{N}_{ACAB} + \mathbf{N}_{ABCA} + \mathbf{N}_{ACBA}),
\]
(B.4)
where \( \dot{E}_{AB} \) are evaluated at \( t_{n+1} \). From this, \( \dot{\mathbf{E}}_{n+1} \) could be extracted to produce the sixth-order tensor \( \frac{\partial \mathbf{\Xi}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}} \); however, it is simpler to obtain the first term in the consistent tangent modulus (B.1) by replacing \( \dot{\mathbf{E}}_{n+1} \) with the tensor of interest \( \mathbf{\Xi}_{n+1}^{(m)} \) (that is, \( \frac{\partial \mathbf{\Xi}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}} = \dot{\mathbf{E}}_{AB} \rightarrow \mathbf{\Xi}_{n+1}^{(m)} \)). To obtain the second term in the consistent tangent modulus, it remains to calculate \( \frac{\partial \mathbf{\Xi}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}} \). From the stress update equation, (5.2.14) or (5.2.23), and corresponding trial stress or strain definition, (5.2.11) or (5.2.16), the functional dependence of the generalized stress is
\[
\mathbf{S}_{n+1}^{(m)} = \frac{\partial \mathbf{S}_{n+1}^{(m)}}{\partial \mathbf{E}_{n+1}^{(m)}} (\Delta \lambda_{n+1}, \mathbf{N}_{n+1}).
\]
(B.5)
Similarly, the growth variable update equations, (5.2.13) or (5.2.22), indicate the functional dependencies

\[ N_{n+1} = \hat{N}_{n+1}(E^{(m)}_{n+1}, E_{g,n+1}, \alpha_{n+1}) \]
\[ E_{g,n+1} = \hat{E}_{g,n+1}(\Delta \lambda_{n+1}, N_{n+1}) \]
\[ \alpha_{n+1} = \hat{\alpha}_{n+1}(\Delta \lambda_{n+1}, N_{n+1}). \]  

(B.6)

Thus,

\[ \frac{\partial S^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} = \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial N^{(m)}_{n+1}} \left( \frac{\partial \hat{N}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{E}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} \right) + \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial \alpha^{(m)}_{n+1}} \frac{\partial \alpha^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} \frac{\partial \Delta \lambda^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}}. \]

(B.7)

The terms readily calculated from the stress update equations, (5.2.14) or (5.2.23), and the definition of the normal vector, (5.1.5), are

\[ \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} = C, \quad \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial N^{(m)}_{n+1}} = -\gamma_n \Delta \lambda_{n+1} |C|, \quad \frac{\partial \hat{N}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} = -\frac{\partial \hat{N}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} = \left( \frac{\mathcal{I} - N^{(m)}_{n+1} \otimes N^{(m)}_{n+1}}{||\Sigma^{(m)}_{n+1}||} \right) C, \]

\[ \frac{\partial \hat{N}^{(m)}_{n+1}}{\partial \alpha^{(m)}_{n+1}} = \left( \frac{\mathcal{I} - N^{(m)}_{n+1} \otimes N^{(m)}_{n+1}}{||\Sigma^{(m)}_{n+1}||} \right) C, \]

\[ \frac{\partial \hat{S}^{(m)}_{n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} = -\gamma_n CN_{n+1}. \]  

(B.8)

The remaining unknowns are \( \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial E^{(m)}_{n+1}}, \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}}, \) and \( \frac{\partial \Delta \lambda^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} \). Two equations for the unknown terms may be generated using (B.6)_{2,3},

\[ \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial E^{(m)}_{n+1}} = \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial N^{(m)}_{n+1}} \left( \frac{\partial \hat{N}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{N}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} \right) + \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial \alpha^{(m)}_{n+1}} \frac{\partial \alpha^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} \frac{\partial \Delta \lambda^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}}, \]

(B.9)

and

\[ \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} = \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial N^{(m)}_{n+1}} + \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial \alpha^{(m)}_{n+1}} \frac{\partial \alpha^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}} + \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} \frac{\partial \Delta \lambda^{(m)}_{n+1}}{\partial E^{(m)}_{n+1}}, \]

(B.10)

where,

\[ \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial N^{(m)}_{n+1}} = \gamma_n \Delta \lambda_{n+1} \mathcal{I}, \quad \frac{\partial \hat{E}^{(m)}_{g,n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} = \gamma_n N^{(m)}_{n+1}, \]
\[ \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial N^{(m)}_{n+1}} = \gamma_n a(\kappa^{(m)}_{n+1}) \Delta \lambda_{n+1} \mathcal{I}, \quad \frac{\partial \hat{\alpha}^{(m)}_{n+1}}{\partial \Delta \lambda^{(m)}_{n+1}} = \gamma_n [a(\kappa^{(m)}_{n+1}) + \sqrt{\frac{2}{3}} a'(\kappa^{(m)}_{n+1}) \Delta \lambda_{n+1}] N^{(m)}_{n+1}. \]  

(B.11)
The derivative of (5.2.9) yields the third equation,
\[
N_{n+1} \cdot \left[ \frac{\partial}{\partial \mathbf{E}_{n+1}^{(m)}} \left( \mathbf{T} - \frac{\partial \mathbf{E}_{g,n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} \right) \right] = H \frac{\partial \kappa_{n+1}}{\partial \Delta \lambda_{n+1}} \frac{\partial \Delta \lambda_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}},
\] (B.12)
where, \( \frac{\partial \kappa_{n+1}}{\partial \Delta \lambda_{n+1}} = \gamma_n \sqrt{\frac{2}{3}} \). Now, (B.9), (B.10), and (B.12) are three tensor equations to be solved for the three unknowns. A lengthy calculation produces the solution,
\[
\frac{\partial \Delta \lambda_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} = \frac{CN_{n+1} - N_{n+1} \cdot (A_1 + CA_2)}{\gamma_n \sqrt{\frac{2}{3}} H + N_{n+1} \cdot (A_1 + CA_2)}
\]
where,
\[
\frac{\partial \alpha_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} = A_1 + A_1 \otimes \frac{\partial \Delta \lambda_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}},
\]
\[
\frac{\partial \mathbf{E}_{g,n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} = A_2 + A_2 \otimes \frac{\partial \Delta \lambda_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}},
\]
where the fourth-order tensors, second-order tensor, and scalars, respectively, are
\[
A_1 = [A_3 - (I + A_3 C)(I + A_1 A_3 C)^{-1}(I + A_1 A_3)]^{-1}[A_3 C - (I + A_3 C)(I + A_1 A_3 C)^{-1}(A_1 A_3 C)]
\]
\[
A_2 = -(I + A_1 A_3 C)^{-1}(I + A_1 A_3)A_1 + (I + A_1 A_3 C)^{-1}(A_1 A_3 C)
\]
\[
A_3 = \gamma_n \Delta \lambda_{n+1} \left( \frac{I - N_{n+1} \otimes N_{n+1}}{|| \Sigma_{n+1} ||} \right)
\]
\[
A_1 = [A_3 - (I + A_3 C)(I + A_1 A_3 C)^{-1}(I + A_1 A_3)]^{-1}[\gamma_n N_{n+1} - (I + A_3 C)(I + A_1 A_3 C)^{-1}(\gamma_n A_2 N_{n+1})]
\]
\[
A_2 = -(I + A_1 A_3 C)^{-1}(I + A_1 A_3)A_2 + (I + A_1 A_3 C)^{-1}(\gamma_n A_2 N_{n+1})
\]
\[
A_1 = 1 + a(\kappa_{n+1})
\]
\[
A_2 = A_1 + \gamma_n \sqrt{\frac{2}{3}} a'(\kappa_{n+1}) \Delta \lambda_{n+1}.
\] (B.14)

Substitute (B.13) into (B.7) which then, with (B.4) yields the consistent tangent modulus (B.1). Depending on the formulation used in the finite element method, the referential algorithmic tangent modulus \( \frac{\partial \kappa_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} \) may need to be pushed forward to the current algorithmic tangent modulus \( C_{n+1} \). The push-forward is achieved through the transformation, given in component form, as
\[
(C_{n+1})_{ijkl} = J^{-1} F_{iA} F_{jB} F_{kC} F_{lD} \left( \frac{\partial \kappa_{n+1}}{\partial \mathbf{E}_{n+1}^{(m)}} \right)_{ABCD},
\] (B.15)
where \( F_{iA} \) are the components of the deformation gradient \( F \) [55, Section 3.4].
Material growth in thermoelastic continua

\[ \kappa(X) = \chi(X,t) \]

Figure 1: Reference and current configurations for a growing continuum.
Figure 2: Multiplicative decomposition of the deformation gradient.
Material growth in thermoelastic continua

$$S_0, \theta_0$$

$$S_g \partial S_g$$

$$S_r \partial S_r$$

$$S_0 \partial S_0$$

$$(S_0, \theta_0)$$

Figure 3: Three sets and corresponding hypersurfaces in stress-temperature space.
Figure 4: Schematic representation of the three-surface activation model.
Material growth in thermoelastic continua

\[
\partial S_{g,n} + 1 S_n(m) T R n+1 S_n(m)
\]

Figure 5: Return-map to growth activation surface.
Figure 6: Return-map to resorption activation surface.
Material growth in thermoelastic continua

Figure 7: Kinematic growth tension/compression cycle.
Figure 8: Constitutive growth tension/compression cycle.
Figure 9: Coupled kinematic and constitutive growth tension/compression cycle.
Figure 10: Artery growth: initial mesh.
Figure 11: Artery growth: relative density change.
Figure 12: Bone adaptation: equivalent Mises stress.
Figure 13: Bone adaptation: relative density change.
Figure 14: Bone adaptation: equivalent Mises stress on interior plane.
Material growth in thermoelastic continua

Figure 15: Bone adaptation: relative density change on interior plane.