A Continuum Theory of Surface Growth

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Abstract

A continuum theory of surface growth in deformable bodies is presented. The theory employs a decomposition of the deformation and growth processes, which leads to a well-posed set of governing equations. It is argued that an evolving reference configuration is required to track the material points in the body. The balance laws are formulated with respect to a non-inertial frame of reference, which is used to track the motion of the body. A one-dimensional example problem is included to showcase the predictive capacity of the theory.

Keywords: Surface growth; cell motility; continuum mechanics; incremental formulation.

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1 Introduction

Continua may undergo volumetric or surface growth. The two extreme cases of the former occur when the density changes at constant volume or when the volume changes at constant density. In either case, the effects are strictly volumetric, in the sense that all growth is effected by changes to the bulk (as opposed to the surface) of the body. Surface growth occurs when material is added to or removed from the surface of the body. To date, the literature of growth mechanics is overwhelmingly concerned with volumetric growth [1, 2, 3, 4, 5].

Surface growth has received very little attention despite the fact that it plays an important role in the overall growth of hard tissues (e.g., teeth, shells) and in the motility of cells. Motility may be effected in certain classes of cells by "treadmilling", which is the addition (resp. removal) of actin monomers at the leading (resp. trailing) edge of the cell, while the bulk of the cell adheres to a substrate. The net effect of treadmilling is the apparent motion of the cell. One-dimensional continuum models of cell motility are proposed by Gracheva and Othmer [6] and Larripa and Mogilner [7]. A two-dimensional model of the same process is suggested by Rubinstein *et al.* [8]. While all these models essentially concern surface growth of the actin network, no attempt is made by the authors to frame their approaches within a general theory of surface growth. On the continuum front, Skalak *et al.* [9, 10] describe models of the kinematics of surface growth for biological tissues without formulating any corresponding balance laws. A preliminary attempt to relate surface growth kinematics to kinetics is contained in the work of Athesian [11], which also recognizes the implications of surface growth on the nature of reference configurations. A review of surface growth kinematics is contained in [12, Section 2].

The objective of this paper is to develop a general three-dimensional continuum theory that can be applied to a broad range of surface growth problems. The three salient features of the proposed theory are: the kinematic representation of surface growth by means of a surface growth velocity field; the decomposition of the motion into growth-free and growthonly parts; and the formulation of the balance laws with respect to a non-inertial frame that depends on the kinematics of growth. As formulated, the theory enables tracking of the deformation relative to an evolving reference configuration which accounts in an incremental manner for the addition and removal of material through the surface. A one-dimensional example is also included to demonstrate the features of the theory in an analytically tractable setting. The paper is organized in two major sections: the first (Section 2) concerns the development of the theory, including the kinematic description and the balance laws, while the second (Section 3) describes a one-dimensional initial/boundary-value problem and documents the main properties of its solution. Concluding remarks are offered in Section 4.

2 Theory

2.1 Kinematics

Let the body \mathscr{B}^t be defined as a collection of particles at time t. Unlike conventional continua, the explicit time designation is essential here due to the possibility of growth. Each material particle P at time t is mapped into the Euclidean point space \mathcal{E}^3 by way of $\bar{\chi}^t : (P,t) \in \mathscr{B}^t \times \mathbb{R} \mapsto \mathcal{E}^3$. The map $\bar{\chi}^t$ will be referred to as the configuration mapping of \mathscr{B}^t at time t. Each point $X \in \mathcal{E}^3$ occupied by a particle at time t is uniquely associated with a vector \mathbf{X} relative to a fixed origin O in the Euclidean vector space E^3 . The image of $\bar{\chi}^t$, denoted by \mathcal{R}^t , is the configuration of the body at time t, and is assumed to be open and to possess a smooth orientable boundary $\partial \mathcal{R}^t$ with outward unit normal $\mathbf{N}(\mathbf{X}, t)$.

It is customary in continuum mechanics to describe the motion of a solid body relative to a fixed reference configuration, frequently taken to be the configuration of the body at some fixed time t_0 . This poses a challenge in the present context, since, when undergoing surface growth, the body may include particles at time t that did not exist at some earlier time t_0 . Conversely, the body at time t may have resorbed, such that material points at time t_0 may no longer be in existence at time t. The difficulty in considering a timevarying set of material particles is dealt with here by continuously updating the reference configuration and decoupling surface growth from pure deformation by imposing surface growth on an intermediate configuration induced by pure deformation. To this end, define a deformation map $\chi_d : \mathcal{R}^{\tau} \times \mathbb{R} \mapsto E^3$, which takes points of the reference configuration \mathcal{R}^{τ} at time τ to an intermediate configuration $\tilde{\mathcal{R}}^{\tau+t}$ at time $\tau + t$, such that

$$\tilde{\mathbf{x}} = \boldsymbol{\chi}_d(\mathbf{X}, t; \tau) = \boldsymbol{\chi}_d^{\tau}(\mathbf{X}, t) .$$
(1)

The deformation map χ_d^{τ} is assumed to be invertible for a fixed t and characterizes the deformation of the body at time $\tau + t$ relative to the configuration at time τ by completely ignoring any surface growth processes in the time interval $(\tau, \tau + t]$. Note that χ_d depends on both τ and t, albeit in distinct manners. Indeed, the former identifies the time at which the reference configuration is defined, while the latter denotes the advancement of time to

the current configuration. In conventional continuum mechanics, it is customary to not include in the argument list of the motion the time at which the reference configuration is defined. This practice is not followed here, owing to the fact that, unlike conventional continuum mechanics, the reference configuration is evolving with time. The notational convention adopted for the argument list of χ_d in $(1)_1$ is intended to underline that the variable(s) to the left of the semicolon may only be defined after the variable(s) to the right. The velocity \mathbf{v}_d associated with the deformation map can be defined on $\tilde{\mathcal{R}}^{\tau+t}$ at time $\tau + t$ as

$$\mathbf{v}_d = \frac{\partial \boldsymbol{\chi}_d \left(\mathbf{X}, t; \tau \right)}{\partial t} \ . \tag{2}$$

The motion χ_d is clearly material for all particles that are persistently in existence during the time interval $(\tau, \tau + t]$.

To account for surface growth, define a growth map $\chi_g : \partial \tilde{\mathcal{R}}^{\tau+t} \mapsto M^2$ from the boundary of the intermediate configuration to a two-dimensional manifold M^2 , such that the position vector of any boundary point of the current configuration $\mathcal{R}^{\tau+t}$ at time $\tau + t$ can be expressed as

$$\mathbf{x} = \boldsymbol{\chi}_g(\tilde{\mathbf{x}}; \tau, t) = \boldsymbol{\chi}_g^{\tau, t}(\tilde{\mathbf{x}}) .$$
(3)

The growth map $\chi_g^{\tau+t}$ is assumed to be a local diffeomorphism between the manifolds $\partial \tilde{\mathcal{R}}^{\tau+t}$ and $\partial \mathcal{R}^{\tau+t}$. Here, the map $\chi_g^{\tau,t}$ fully characterizes the growth process in the time interval $(\tau, \tau+t]$. The map χ_g is generally not material, as it merely tracks the evolution of the boundary of the body due to surface growth. It is useful in the ensuing developments to define a mean rate of growth \mathbf{v}_g on $\partial \tilde{\mathcal{R}}^{\tau+t}$ during the time interval $(\tau, \tau+t]$ as

$$\mathbf{v}_g = \frac{1}{t} (\mathbf{x} - \tilde{\mathbf{x}}) = \frac{1}{t} \left(\boldsymbol{\chi}_g^{\tau, t} \left(\tilde{\mathbf{x}} \right) - \tilde{\mathbf{x}} \right) \,. \tag{4}$$

The normal component $\mathbf{v}_g \cdot \tilde{\mathbf{n}}$ of the growth velocity quantifies the rate at which material is added $(\mathbf{v}_g \cdot \tilde{\mathbf{n}} > 0)$ or removed $(\mathbf{v}_g \cdot \tilde{\mathbf{n}} < 0)$ at the surface $\partial \tilde{\mathcal{R}}^{\tau+t}$. No surface growth (or resorption) takes place when $\mathbf{v}_g \cdot \tilde{\mathbf{n}} = 0$. Here, $\tilde{\mathbf{n}}$ denotes the outward unit normal to the surface $\partial \tilde{\mathcal{R}}^{\tau+t}$.

Taking into account (1) and (3), the apparent motion $\chi_a : \partial \mathcal{R}^{\tau} \times \mathbb{R} \mapsto M^2$ of points on the boundary $\partial \mathcal{R}^{\tau}$ is

$$\boldsymbol{\chi}_a = \boldsymbol{\chi}_g^{\tau,t} \circ \boldsymbol{\chi}_d^{\tau} \ . \tag{5}$$

Once the boundary $\partial \mathcal{R}^{\tau+t}$ is determined, the region $\mathcal{R}^{\tau+t}$ occupied by the body at time $\tau + t$ is defined by

$$\mathcal{R}^{\tau+t} = \operatorname{int} \left(\partial \mathcal{R}^{\tau+t} \right) , \qquad (6)$$

where "int" denotes the interior of an orientable surface. The regions $\partial \tilde{\mathcal{R}}^{\tau+t}$ and $\partial \mathcal{R}^{\tau+t}$ defined by means of the decomposition (5) are depicted in Figure 1.

The decomposition (5) is an essential feature of the proposed theory. Indeed, this decomposition enables the definition of the region $\mathcal{R}^{\tau+t}$ for bodies undergoing surface growth. As will be established later in this section, the decomposition (5) also permits the distinction between growth regions and material regions in the time interval $(\tau, \tau + t]$. Note that the order of the composition of χ_g and χ_d is important in the ensuing developments. This is because admitting that the body first undergoes a pure deformation followed by growth enables the definition of the maps χ_d and χ_g on $\partial \mathcal{R}^{\tau}$ and $\partial \tilde{\mathcal{R}}^{\tau+t}$, which are simple to motivate. Specifically, one may take χ_d to apply to the body in $(\tau, \tau + t]$ by assuming that growth is suppressed. This is followed by χ_g , which effects all surface growth without deformation. The alternative of imposing surface growth first on the configuration $\partial \mathcal{R}^{\tau}$, while not in any sense erroneous, is conceptually less appealing. Indeed, if surface growth/resorption were to be effected first, then boundary conditions would need to be enforced on material points that come to existence (or reach the boundary of the body) simultaneously with the application of these boundary conditions.

The preceding development depends crucially on the elapsed time t between the reference and the current configuration. Unlike conventional continuum mechanics, some restrictions need to be placed on the magnitude of t. Specifically, t should be much smaller than the characteristic growth time t_c^{τ} , defined as

$$t_c^{\tau} = \frac{L^{\tau}}{\|\mathbf{v}_g\|_{\partial \tilde{\mathcal{R}}^{\tau+t}}} , \qquad (7)$$

where L^{τ} is a characteristic measure of length for the reference configuration, defined as

$$L^{\tau} = \min(L_1^{\tau}, L_2^{\tau}, L_3^{\tau})$$
,

by means of appropriately chosen characteristic lengths L_i^{τ} , i = 1, 2, 3, for each spatial dimension. Also, $\|\cdot\|_{\partial \tilde{\mathcal{R}}^{\tau+t}}$ in (7) denotes the L_2 -norm on $\partial \tilde{\mathcal{R}}^{\tau+t}$. This restriction ensures that some of the particles in \mathcal{R}^{τ} survive in $\mathcal{R}^{\tau+t}$, therefore the deformation map χ_d is well-defined. Moreover, t should be small enough so that the sign of $\mathbf{v}_g \cdot \tilde{\mathbf{n}}$ does not change between times τ and $\tau + t$ when tracking any point $\tilde{\mathbf{x}} \in \partial \tilde{\mathcal{R}}^{\tau+t}$ associated with a given point \mathbf{X} in the reference configuration. This restriction guarantees that the time resolution is sufficiently fine to capture all growth/resorption processes within the time interval. Lastly, t should be small enough for the continuous interaction between deformation and surface growth to be satisfactorily represented in $(\tau, \tau + t]$ by the composition of χ_d^{τ} and $\chi_g^{\tau,t}$ in (5). A lower bound for t is provided by requiring that the total linear growth in $(\tau, \tau + t]$ exceed the characteristic length of a typical molecular constituent of the growing material (e.g., the length of an actin monomer). This restriction underlines the discrete nature of surface growth physics.

The foregoing kinematic development implies that the current configuration $\mathcal{R}^{\tau+t}$ is comprised of materials occupying two disjoint open sets in E^3 : a material region $\mathcal{M}^{\tau+t}$ consisting of material particles at $\tau + t$ that also existed at time τ , and a growth region $\mathcal{G}^{\tau+t}$, none of the material particles of which existed at time τ . In addition, define a surface $\sigma^{\tau+t}$, which corresponds to the interface between the material and growth regions, as

$$\sigma^{\tau+t} = \overline{\mathcal{M}}^{\tau+t} \cap \overline{\mathcal{G}}^{\tau+t} .$$
(8)

Now, the current configuration may be expressed mathematically as

$$\mathcal{R}^{\tau+t} = \mathcal{M}^{\tau+t} \cup \mathcal{G}^{\tau+t} \cup \sigma^{\tau+t} .$$
(9)

The ablated region $\mathcal{A}^{\tau+t}$ at time $\tau + t$ relative to time τ is also readily determined as

$$\mathcal{A}^{\tau+t} = \tilde{\mathcal{R}}^{\tau+t} \setminus \mathcal{R}^{\tau+t} .$$
(10)

It is clear that the growth region $\mathcal{G}^{\tau+t}$ needs to be endowed with density, velocity and deformation information at time $\tau + t$. The preceding fields are already defined on the interface $\sigma^{\tau+t}$ and will now be extended to the domain of $\mathcal{G}^{\tau+t}$. Generally, the density of the material in the growth region may be different from the density of the original material, hence it is not necessary to require continuity of ρ across $\sigma^{\tau+t}$. In case the physical process of growth necessitates continuity, then a smooth extension of ρ from the boundary region $\sigma^{\tau+t}$ to the open set $\mathcal{G}^{\tau+t}$ can be effected. In the context of Sobolev spaces, such an extension can be formally shown to exist under certain conditions on the smoothness of the boundary and the distribution of ρ on $\sigma^{\tau+t}$. For instance, it is known that a constant ρ on $\sigma^{\tau+t}$ can be smoothly extended to $\mathcal{G}^{\tau+t}$ using a version of the classical trace theorem [13, Theorem 5.6]. Likewise, smooth extensions from non-constant fields on $\sigma^{\tau+t}$ to the full boundary $\partial \mathcal{G}^{\tau+t}$ of $\mathcal{G}^{\tau+t}$ have been constructed for specific geometries of $\mathcal{G}^{\tau+t}$ [14, Lemma A.1]. Once a smooth ρ is defined throughout $\partial \mathcal{G}^{\tau+t}$, the classical trace theorem [15, Theorem 1.5.1.2] guarantees the existence of a smooth ρ in $\mathcal{G}^{\tau+t}$. An alternative constructive approach for polynomial extensions is discussed in [16]. Non-polynomial smooth extensions can be obtained by formulating a mixed elliptic boundary-value problem in $\mathcal{G}^{\tau+t}$ with Dirichlet boundary conditions on $\sigma^{\tau+t}$ and homogeneous Neumann boundary conditions elsewhere.

The extension of the deformation gradient into $\mathcal{G}^{\tau+t}$ presents an obvious challenge. Indeed, the deformation gradient, by definition, is taken relative to a reference configuration and, in conventional continuum mechanics, this configuration is common to all material points at a given time. Here, such a deformation gradient cannot be defined, since the particles in $\mathcal{G}^{\tau+t}$ did not come into existence until after time τ . Therefore, by necessity, the deformation gradient at time $\tau + t$ is written relative to multiple configurations. In particular, the deformation gradient in $\mathcal{M}^{\tau+t}$ may be defined by concession relative to \mathcal{R}^{τ} , while in $\mathcal{G}^{\tau+t}$ it can only be defined relative to $\mathcal{G}^{\tau+t}$ itself, hence it is equal to the identity tensor in this region. As a result, the deformation gradient is not uniquely defined on the interface $\sigma^{\tau+t}$ between $\mathcal{M}^{\tau+t}$ and $\mathcal{G}^{\tau+t}$. In summary, one may write

$$\mathbf{F}_{\tau}^{\tau+t} = \begin{cases} \frac{\partial \boldsymbol{\chi}_{d}^{\tau}}{\partial \mathbf{X}} & \text{on } \mathcal{M}^{\tau+t} \\ \mathbf{I} & \text{on } \mathcal{G}^{\tau+t} \\ \text{undefined } \text{on } \mathcal{A}^{\tau+t} \\ \text{undefined } \text{on } \sigma^{\tau+t} \end{cases}$$
(11)

Clearly, the calculation of the deformation gradient for any newly formed material particle is performed relative to the position occupied by that material particle at or after the time of its creation. The latter may vary for different material particle in the body.

Extensions of other kinematic quantities from $\sigma^{\tau+t}$ to $\mathcal{G}^{\tau+t}$ can be effected, as argued for the case of the mass density. However, different continuity assumptions may be required depending on the nature of each quantity. For instance, the velocity \mathbf{v}_d should be extended continuously from $\sigma^{\tau+t}$ to $\mathcal{G}^{\tau+t}$ to ensure that the growth region remains attached to the rest of the body along $\sigma^{\tau+t}$.

The kinematic model defined herein includes as a special case the non-growing deforming continuum, which corresponds to the case of χ_g being the identity map on $\partial \tilde{\mathcal{R}}^{\tau+t}$ and χ_d being a non-trivial map on $\partial \mathcal{R}^{\tau}$. Likewise, a growing rigid continuum corresponds to the case of χ_g being a non-trivial map on $\partial \tilde{\mathcal{R}}^{\tau+t}$ and χ_d being a global rotation map on \mathcal{R}^{τ} . In the former case, the motion of the body is purely material, while in the latter the (apparent) motion is exclusively due to the addition (resp. removal) of material points to (resp. from) the surface of the body. As already argued, the meaning of a reference configuration is substantially altered in the presence of surface growth. For instance, in the case of pure growth, while the body is experiencing a shape-altering "motion", it never actually leaves its reference configuration.

2.2 Balance Laws

Balance laws are defined relative to a given frame of reference. Following Truesdell and Toupin [17, Sections 196-197], a frame of reference is termed inertial if one may express Euler's two laws relative to it in the canonical form

$$\dot{\mathbf{G}} = \mathbf{F} \quad , \quad \dot{\mathbf{H}}^O = \mathbf{M}^O \; . \tag{12}$$

Here, **G** is the linear momentum of the body (or any part of it), **F** is the resultant external force, \mathbf{H}^{O} is the angular momentum about a fixed point O, and \mathbf{M}^{O} is the resultant moment of the external forces about the point O. When using a non-inertial frame, material time derivatives on the left-hand sides of the two laws in (12) are replaced by frame-specific time derivatives and additional external forces are included on the right-hand sides to account for the flow of momenta across non-material boundaries.

For the purposes of this paper, a non-inertial frame of reference is employed to formulate the balance laws. In particular, the frame of reference has a motion that is specified independently of the body, via the velocity $\mathbf{v}_f(\mathbf{x}, t; \tau)$. The goal is to have the frame of reference track the apparent motion of the body, as described by (5), including the effects of both deformation and growth. To this end, recall that the particle velocity in $\tilde{\mathcal{R}}^{\tau+t}$ with respect to any fixed frame is equal to \mathbf{v}_d . The frame velocity \mathbf{v}_f is chosen on the boundary $\partial \tilde{\mathcal{R}}^{\tau+t}$ such that

$$(\mathbf{v}_f - \mathbf{v}_d) \cdot \tilde{\mathbf{n}} = \mathbf{v}_g \cdot \tilde{\mathbf{n}} . \tag{13}$$

This ensures that the non-inertial frame tracks the evolving boundary of the body, in the sense that the coordinates of this boundary relative to the moving frame remain fixed. The condition in (13) implies that there is no restriction on the tangential component of \mathbf{v}_f . Likewise, the extension of \mathbf{v}_f to $\tilde{\mathcal{R}}^{\tau+t}$ is effected by appealing to the standard trace theorem. No physical interpretation is assigned to \mathbf{v}_f in the interior of the body, beyond the fact that it represents the velocity of a non-inertial frame attached to the growing boundary.

A global statement of balance of mass relative to the inertial frame can be readily derived by recalling the identity

$$\frac{d}{dt} \int_{\tilde{\mathcal{P}}} \rho \, dv = \frac{\partial}{\partial t} \int_{\tilde{\mathcal{P}}} \rho \, dv + \int_{\partial \tilde{\mathcal{P}}} \rho \mathbf{v}_d \cdot \tilde{\mathbf{n}} \, da , \qquad (14)$$

where $\frac{d}{dt}$ denotes the material time derivative and $\tilde{\mathcal{P}} \subset \tilde{\mathcal{R}}^{\tau+t}$ is an arbitrary fixed region of the body with boundary $\partial \tilde{\mathcal{P}}$. A corresponding identity can be similarly derived for the rate of change of mass relative to the non-inertial frame as

$$\frac{d_f}{dt} \int_{\tilde{\mathcal{P}}_f} \rho \, dv_f = \frac{\partial}{\partial t} \int_{\tilde{\mathcal{P}}_f} \rho \, dv_f + \int_{\partial \tilde{\mathcal{P}}_f} \rho \mathbf{v}_f \cdot \tilde{\mathbf{n}} \, da_f , \qquad (15)$$

in an arbitrary fixed region $\tilde{\mathcal{P}}_f \subset \tilde{\mathcal{R}}^{\tau+t}$ with boundary $\partial \tilde{\mathcal{P}}_f$. Here, $\frac{d_f}{dt}$ denotes the time derivative keeping the coordinates of the non-inertial frame fixed. Recalling the identities in (14) and (15), setting $\tilde{\mathcal{P}}_f = \tilde{\mathcal{P}}$, and imposing conservation of mass in the material region $\tilde{\mathcal{P}}$, leads to

$$\frac{d_f}{dt} \int_{\tilde{\mathcal{P}}} \rho \, dv = \int_{\partial \tilde{\mathcal{P}}} \rho \left(\mathbf{v}_f - \mathbf{v}_d \right) \cdot \tilde{\mathbf{n}} \, da , \qquad (16)$$

which is the desired integral statement of mass balance. A local form may be deduced from (16) by using standard versions of the transport, divergence, and localization theorems, and reads

$$\frac{d_f \rho}{dt} + \rho \operatorname{div} \mathbf{v}_d = \operatorname{grad} \rho \cdot (\mathbf{v}_f - \mathbf{v}_d) .$$
(17)

The preceding derivation follows closely the steps used in obtaining the equations of motion in Arbitrary Lagrangian-Eulerian formulations of conventional continuum mechanics [18].

In the special case $\mathcal{P} = \tilde{\mathcal{R}}^{\tau+t}$, namely when considering the complete body in the intermediate configuration, the total rate of change of mass M is given by

$$\frac{d_f M}{dt} = \frac{d_f}{dt} \int_{\tilde{\mathcal{R}}^{\tau+t}} \rho \, dv = \int_{\partial \tilde{\mathcal{R}}^{\tau+t}} \rho \mathbf{v}_g \cdot \tilde{\mathbf{n}} \, da , \qquad (18)$$

where use is made of (13) and (16).

Statements for the balance of linear and angular momentum in the growing body can be derived in complete analogy to the derivation of mass balance. For linear momentum, the integral statement of balance takes the form

$$\frac{d_f}{dt} \int_{\tilde{\mathcal{P}}} \rho \mathbf{v}_d \, dv = \int_{\tilde{\mathcal{P}}} \rho \mathbf{b} \, dv + \int_{\partial \tilde{\mathcal{P}}} \mathbf{t} \, da + \int_{\partial \tilde{\mathcal{P}}} \rho \mathbf{v}_d \left[(\mathbf{v}_f - \mathbf{v}_d) \cdot \tilde{\mathbf{n}} \right] \, da \;, \tag{19}$$

where **b** is the body force per unit mass and **t** the surface traction of $\partial \tilde{\mathcal{P}}$. The corresponding local form of linear momentum balance is

$$\rho \frac{d_f \mathbf{v}_d}{dt} = \rho \mathbf{b} + \operatorname{div} \mathbf{T} + (\operatorname{grad} \mathbf{v}_d) \rho \left(\mathbf{v}_f - \mathbf{v}_d\right) , \qquad (20)$$

where **T** is the Cauchy stress tensor. Again, for the case $\tilde{\mathcal{P}} = \tilde{\mathcal{R}}^{\tau+t}$, one may express the rate of change of linear momentum for the whole body as

$$\frac{d_f}{dt} \int_{\tilde{\mathcal{R}}^{\tau+t}} \rho \mathbf{v}_d \, dv = \int_{\tilde{\mathcal{R}}^{\tau+t}} \rho \mathbf{b} \, dv + \int_{\partial \tilde{\mathcal{R}}^{\tau+t}} \mathbf{t} \, da + \int_{\partial \tilde{\mathcal{R}}^{\tau+t}} \rho \mathbf{v}_d \left(\mathbf{v}_g \cdot \tilde{\mathbf{n}} \right) da \,. \tag{21}$$

Angular momentum balance for the growing body can be expressed in the non-inertial frame as

$$\frac{d_f}{dt} \int_{\tilde{\mathcal{P}}} \mathbf{x} \times \rho \mathbf{v} \, dv = \int_{\tilde{\mathcal{P}}} \mathbf{x} \times \rho \mathbf{b} \, dv + \int_{\tilde{\mathcal{P}}} \left(\mathbf{e} \left[\mathbf{T}^T \right] + \mathbf{x} \times \operatorname{div} \mathbf{T} \right) \, dv + \int_{\tilde{\mathcal{P}}} \operatorname{div} \left(\left(\mathbf{x} \times \rho \mathbf{v}_d \right) \otimes \left(\mathbf{v}_f - \mathbf{v}_d \right) \right) \, dv \,, \quad (22)$$

where $\mathbf{e}[\mathbf{A}]$ denotes the alternator tensor acting on a second-order tensor \mathbf{A} . Expanding the left-hand side of (22) and taking into account (17) and (20) leads to

$$\int_{\tilde{\mathcal{P}}} \frac{d_f \mathbf{x}}{dt} \times \rho \mathbf{v}_d \, dv = \int_{\tilde{\mathcal{P}}} \left(\mathbf{e} \left[\mathbf{T}^T \right] + \mathbf{v}_f \times \rho \mathbf{v}_d \right) \, dv \;. \tag{23}$$

Since $\frac{d_f \mathbf{x}}{dt} = \mathbf{v}_f$, balance of angular momentum yields the usual symmetry condition for the Cauchy stress tensor **T**.

The Cauchy stress **T** is defined everywhere in the intermediate configuration at time $\tau + t$, including the growth region that has come into existence by time τ (*i.e.*, the growth region that existed by the end of the previous time interval). In this region, the stress at time $\tau + t$ may depend on the deformation gradient relative to the configuration at τ , as well as on other variables.

3 An Initial/Boundary-Value Problem

3.1 Problem formulation

The strong form of the surface growth problem may be stated as follows: determine the density $\rho(\tilde{\mathbf{x}}, t; \tau)$: $\tilde{\mathcal{R}}^{\tau+t} \times \mathcal{I} \mapsto \mathbb{R}$ and displacement $\mathbf{u}(\tilde{\mathbf{x}}, t; \tau)$: $\tilde{\mathcal{R}}^{\tau+t} \times \mathcal{I} \mapsto \mathbb{R}^3$ that satisfy (17) and (20), subject to the initial conditions

$$\rho(\tilde{\mathbf{x}}, 0; \tau) = \rho_0(\mathbf{X}) \text{ in } \mathcal{R}^{\tau} ,
\mathbf{u}(\tilde{\mathbf{x}}, 0; \tau) = \mathbf{u}_0(\mathbf{X}) \text{ in } \mathcal{R}^{\tau} ,
\mathbf{v}_d(\tilde{\mathbf{x}}, 0; \tau) = \mathbf{v}_0(\mathbf{X}) \text{ in } \mathcal{R}^{\tau} ,$$
(24)

the boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}}(\tilde{\mathbf{x}}, t; \tau) \text{ on } \tilde{\Gamma}_{u} \times \mathcal{I} ,$$

$$\mathbf{t} = \bar{\mathbf{t}}(\tilde{\mathbf{x}}, t; \tau) \text{ on } \tilde{\Gamma}_{q} \times \mathcal{I} ,$$
(25)

and the extensions of ρ , **u**, and **v**_d into the growth region \mathcal{G} as formulated in Section 2.1. In the above, \mathcal{I} is the time interval $(\tau, T]$, where T is the terminal time of the interval. Also, the domains of the Dirichlet and Neumann boundary conditions (25) satisfy $\tilde{\Gamma}_u \bigcup \tilde{\Gamma}_q = \partial \tilde{\mathcal{R}}^{\tau+t}$. It is emphasized here that the frame velocity \mathbf{v}_f is assumed to be known throughout the domain occupied by the body in its intermediate configuration.

3.2 A One-Dimensional Example

Consider a one-dimensional continuum lying along the X-axis, such that $\mathcal{R}^0 = \{X \in \mathcal{E}^1 \mid A \leq X \leq B\}$, with the body deforming and growing over the time interval $(0, t_1]$ with no body force. Dirichlet boundary conditions that are linear in time are assumed at both end points, namely

$$u(A,t) = \bar{v}_A t \quad , \quad u(B,t) = \bar{v}_B t \; ,$$
 (26)

hence

$$u(A,t_1) = \bar{u}_A = \bar{v}_A t_1 \quad , \quad u(B,t_1) = \bar{u}_B = \bar{v}_B t_1 \; . \tag{27}$$

It follows that the positions of the boundary points in the intermediate configuration are

$$\tilde{x}(A,t_1) = A + \bar{u}_A = \tilde{a} , \quad \tilde{x}(B,t_1) = B + \bar{u}_B = \tilde{b} .$$
 (28)

At the same time, the body is experiencing surface growth, with growth rates $v_g(A) = \bar{v}_{g_A}$ and $v_g(B) = \bar{v}_{g_B}$ that are assumed constant over the time interval $(0, t_1]$. The initial conditions for the problem are that the mass density at time $\tau = 0$ is homogeneous and equal to ρ_0 , while the initial displacement u_0 and velocity v_0 are equal to zero. In this case, and for a given frame velocity v_f , there are two balance equations (corresponding to mass and linear momentum) and two unknowns, namely v_d and ρ .

The frame velocity specified here is consistent with the condition in (13), which, in the present case, reduces to $v_g = v_f - v_d$ at the two boundary points. This implies that the boundary velocities of the frame are specified and equal to

$$v_f(A) = \bar{v}_A + \bar{v}_{g_A} , \quad v_f(B) = \bar{v}_B + \bar{v}_{g_B} .$$
 (29)

A smooth extension of the frame velocity from the boundary to the interior of the domain can be constructed by linear interpolation, such that

$$v_f = v_{f0} + v_{f1} X_f , (30)$$

where, using (27) and (29),

$$v_{f0} = \frac{1}{B-A} \left(\left(\bar{v}_{gA} + \frac{\bar{u}_A}{t_1} \right) B - \left(\bar{v}_{gB} + \frac{\bar{u}_B}{t_1} \right) A \right) ,$$

$$v_{f1} = \frac{1}{B-A} \left(\bar{v}_{gB} + \frac{\bar{u}_B}{t_1} - \bar{v}_{gA} - \frac{\bar{u}_A}{t_1} \right) .$$
(31)

Now, integrating the frame velocity in (30) with respect to time yields an expression for the motion of the frame as

$$\tilde{x}_f = X_f + (v_{f0} + v_{f1}X_f)t . (32)$$

A solution of the governing equations of motion is obtained using a semi-inverse approach. To this end, assume that the material motion in the time interval $(0, t_1]$ is of the form

$$\tilde{x} = X + (v_0 + v_1 X) t , \qquad (33)$$

where v_0 , v_1 are constants to be determined. Therefore, (referential) balance of mass implies that

$$\rho = \rho_0 \left(1 + v_1 t \right)^{-1} \ . \tag{34}$$

Proceeding with enforcement of linear momentum balance, note that, in the absence of body force and with a constitutive law that depends only on the (assumed) homogeneous strain, (20) reduces to

$$\frac{d_f v_d}{dt} = \frac{\partial v_d}{\partial \tilde{x}} \left(v_f - v_d \right) \ . \tag{35}$$

Using (33), the preceding equation can be rewritten as

$$\frac{d_f v_d}{dt} = \frac{v_1}{1 + v_1 t} \left(v_f - v_d \right) . \tag{36}$$

The frame-time derivative of the velocity on the left-hand side of equation (36) is determined to be

$$\frac{d_{f}v_{d}}{dt} = \frac{\partial}{\partial t} \left(v_{0} + v_{1} \left(\frac{\tilde{x}_{f} - v_{0}t}{1 + v_{1}t} \right) \right) \Big|_{X_{f}},
= \frac{\partial}{\partial t} \left(v_{0} + v_{1} (X_{f} + (v_{f0} + v_{f1}X_{f})t)(1 + v_{1}t)^{-1} - v_{1}v_{0}t(1 + v_{1}t)^{-1} \right) \Big|_{X_{f}},
= \frac{v_{1}}{1 + v_{1}t} \left(v_{f} - \frac{v_{1}}{1 + v_{1}t}\tilde{x}_{f} - v_{0} + \frac{v_{1}}{1 + v_{1}t}v_{0}t \right),$$

$$= \frac{v_{1}}{1 + v_{1}t} \left(v_{f} - (v_{0} + v_{1}X) \right),
= \frac{v_{1}}{1 + v_{1}t} \left(v_{f} - (v_{0} + v_{1}X) \right),$$
(37)

where use is made of (32) and (33). Also, in (37) the notation $(\cdot)|_{X_f}$ signifies that a partial time derivative is taken keeping the frame coordinates fixed.

In summary, a comparison of the results in (36) and (37) shows that the assumed motion (33) satisfies linear momentum balance. Lastly, note that the constants v_0 and v_1 can be determined in terms of the Dirichlet boundary conditions (27) and the end time t_1 as

$$v_{0} = \frac{1}{(B-A)t_{1}} (B\bar{u}_{A} - A\bar{u}_{B}) ,$$

$$v_{1} = \frac{1}{(B-A)t_{1}} (\bar{u}_{B} - \bar{u}_{A}) .$$
(38)

The final configuration \mathcal{R}^{t_1} is now readily determined to occupy the region (a, b), where

$$a = \tilde{a} + \bar{v}_{gA}t_1 ,$$

$$b = \tilde{b} + \bar{v}_{gB}t_1 ,$$
(39)

with boundary $\partial \mathcal{R}^{t_1} = \{a, b\}.$

A sketch of this motion is shown in Figure 2 for the special case $\bar{v}_A = 0$, $\bar{v}_B > 0$, $\bar{v}_{gA} > 0$ and $\bar{v}_{gB} > 0$. In this case, the left boundary undergoes ablation, while the right boundary experiences surface growth. Note that the ablation of material alters the value of the displacement at the boundary, since the boundary point upon which the Dirichlet condition is originally applied is ablated by the end of the time interval. Also, note that the displacement field exhibits a jump induced by its extension into the growth region.

Next, it is instructive to consider the deformation incurred during an additional finite time interval $(\tau_2, \tau_2 + t_2]$, where $\tau_2 = t_1$ and the reference configuration is updated such that $\mathcal{R}^{\tau_2} = \mathcal{R}^{t_1}$ (therefore, $A^{\tau_2} = a$ and $B^{\tau_2} = b$). Again, for this time interval, Dirichlet boundary conditions are specified for each end of the body in the form

$$u(A^{\tau_2}, t) = \bar{\bar{v}}_A(t - \tau_2) \quad , \quad u(B^{\tau_2}, t) = \bar{\bar{v}}_B(t - \tau_2) \; . \tag{40}$$

Also, no growth is assumed to take place during this time interval, hence the frame velocity coincides with the material velocity.

Adopting, again, a semi-inverse approach, assume that the displacement $u(X, t; \tau_2) = x^{\tau_2+t} - X$ is linear in X for each region \mathcal{M}^{t_1} and \mathcal{G}^{t_1} separately, namely

$$u(X,t;\tau_2) = \begin{cases} \left(v_2^- + v_3^- X\right)(t-\tau_2) , & A^{\tau_2} \le X < C^{\tau_2} , \tau_2 \le t \le \tau_2 + t_2 \\ \left(v_2^+ + v_3^+ X\right)(t-\tau_2) , & C^{\tau_2} < X \le B^{\tau_2} , \tau_2 \le t \le \tau_2 + t_2 \end{cases}, \quad (41)$$

where $C^{\tau_2} = c \in \mathcal{R}^{t_1}$ is the growth surface for the first time interval. The four constants v_2^-, v_3^-, v_2^+ and v_3^+ in (41) need to be determined from the two boundary conditions in (40) and two additional conditions emanating from geometric compatibility and the balance

laws. The first of the latter two conditions is due to the fact that for all times $t \ge \tau_2$ for which the region \mathcal{G}^{t_1} exists, it is considered to be a material region. In particular, this implies that the displacement is continuous at point C^{τ_2} , hence

$$\left(v_{2}^{-}+v_{3}^{-}C^{\tau_{2}}\right)\left(t-\tau_{2}\right) = \left(v_{2}^{+}+v_{3}^{+}C^{\tau_{2}}\right)\left(t-\tau_{2}\right) .$$

$$(42)$$

Linear momentum balance holds true at each interior point of the regions $\mathcal{R}^{\tau_2^-} = \{X : A^{\tau_2} \leq X < C^{\tau_2}\}$ and $\mathcal{R}^{\tau_2^+} = \{X : C^{\tau_2} < X \leq B^{\tau_2}\}$. Indeed, in each of these regions, given that the displacements are linear in the reference coordinates, the deformation gradient is necessarily constant, hence the divergence of the stress vanishes identically. Furthermore, the system experiences no accelerations, as dictated by the assumed displacement in (41). Therefore, linear momentum balance is satisfied in the absence of body forces. To enforce linear momentum balance on the interfacial point C^{τ_2} , it is sufficient (albeit not necessary) for the deformation gradient to be continuous at that point¹. Taking into account (33) and (41), the deformation gradients F^{t-} and F^{t+} of the two regions relative to their respective reference configurations are easily found to be

$$F^{t-} = F^{t-}_{\tau_2} F^{t_1-}_0 = \left(1 + v_3^- (t - \tau_2)\right) (1 + v_1 t_1) , \qquad (43)$$

and

$$F^{t+} = F^{t+}_{\tau_2} = 1 + v^+_3 (t - \tau_2) .$$
(44)

Equating the two deformation gradients in (43) and (44) leads to

$$\left(1 + v_3^-(t - \tau_2)\right)\left(1 + v_1t_1\right) = 1 + v_3^+(t - \tau_2) .$$
(45)

It is important to note that, while the preceding equation imposes continuity (and, more specifically, constancy) of the total deformation gradient in the regions $\mathcal{R}^{\tau_2^-}$ and $\mathcal{R}^{\tau_2^+}$, the deformation gradient relative to the configuration at time τ_2 remains piecewise constant, as implied by (41).

It is now possible to determine the constants v_2^- , v_3^- , v_2^+ and v_3^+ from (42), (45) and the boundary conditions at time $t = \tau_2 + t_2$. The latter can be expressed as

$$\bar{\bar{u}}_A = \bar{\bar{v}}_A t_2 \quad , \quad \bar{\bar{u}}_B = \bar{\bar{v}}_B t_2 \quad , \tag{46}$$

¹It is entirely conceivable that, depending on the particular nature of the constitutive equation for the stress, other solutions of the balance of linear momentum that effect a discontinuity in the deformation gradient exist.

which, with the aid of (41), become

$$(v_2^- + v_3^- A^{\tau_2}) t_2 = \bar{\bar{u}}_A \quad , \quad (v_2^+ + v_3^+ B^{\tau_2}) t_2 = \bar{\bar{u}}_B \; .$$
 (47)

The resulting system yields a solution

$$\begin{bmatrix} v_{2}^{-} \\ v_{3}^{-} \\ v_{3}^{+} \end{bmatrix} = \begin{bmatrix} -\frac{B^{\tau_{2}}\bar{u}_{A} + B^{\tau_{2}}v_{1}t_{1}\bar{u}_{A} + B^{\tau_{2}}v_{1}t_{1}A^{\tau_{2}} - \bar{u}_{B}A^{\tau_{2}} - C^{\tau_{2}}v_{1}t_{1}\bar{u}_{A} - C^{\tau_{2}}v_{1}t_{1}A^{\tau_{2}}}{(A^{\tau_{2}} + C^{\tau_{2}}v_{1}t_{1} - B^{\tau_{2}} - B^{\tau_{2}}v_{1}t_{1})t_{2}} \\ -\frac{C^{\tau_{2}}v_{1}t_{1} + \bar{u}_{A} - \bar{u}_{B} + B^{\tau_{2}}v_{1}t_{1}}{(A^{\tau_{2}} + C^{\tau_{2}}v_{1}t_{1} - B^{\tau_{2}} - B^{\tau_{2}}v_{1}t_{1})t_{2}} \\ -\frac{-B^{\tau_{2}}C^{\tau_{2}}v_{1}t_{1} + B^{\tau_{2}}\bar{u}_{A} + B^{\tau_{2}}v_{1}t_{1}\bar{u}_{A} + B^{\tau_{2}}v_{1}t_{1}A^{\tau_{2}} - \bar{u}_{B}A^{\tau_{2}} - \bar{u}_{B}C^{\tau_{2}}v_{1}t_{1}}{(A^{\tau_{2}} + C^{\tau_{2}}v_{1}t_{1} - B^{\tau_{2}} - B^{\tau_{2}}v_{1}t_{1})t_{2}} \\ -\frac{-C^{\tau_{2}}v_{1}t_{1} + \bar{u}_{A} - \bar{u}_{B} + v_{1}t_{1}\bar{u}_{A} - v_{1}t_{1}\bar{u}_{B} + v_{1}t_{1}A^{\tau_{2}}}{(A^{\tau_{2}} + C^{\tau_{2}}v_{1}t_{1} - B^{\tau_{2}} - B^{\tau_{2}}v_{1}t_{1})t_{2}} \end{bmatrix}$$

$$(48)$$

A plot of the displacement is displayed in Figure 3 for the special case $\bar{\bar{u}}_A = \bar{\bar{u}}_B = 0$. As argued, the total displacement is discontinuous.

It is important to examine whether the discontinuities in the displacement are a salient feature of the formulation or the result of the incremental application of surface growth. To this end, consider the limit of the solution at some fixed time t_f (taken here to be $t_f = 1.0$) as the number of growth time intervals $n = \frac{t_f}{\Delta \tau}$ tends to infinity, where $\Delta \tau$ denotes the (constant) time between distinct growth events. A series of such solutions is depicted in Figures 4–6. The case n = 1 in Figure 4 shows the extension of the displacement field into the growth region after a single time interval, as described earlier. Increasing values of n show a regular change in the displacements over the growth region. Figure 5 shows a detail of the displacement for the case n = 100, clearly illustrating the discontinuities. It is clear from Figure 4 that the jumps in the displacement between neighboring growth regions are reduced as n is increasing, while at the same time the total number of discontinuity points is increasing. The limiting case of the displacement for $n \to \infty$ can be approximately constructed by taking a piecewise linear curve that connects the centers of each discontinuous segment in the case n = 100. This limiting case is plotted versus the case n = 1 in Figure 6 and shows a trend of softening in the displacement with increasing n.

As expected, the displacement across the whole growth region exhibits a negative gradient, reflecting the incremental addition of material particles on the evolving surface of the body. At the same time, balance of linear momentum clearly dictates that the stresses remain non-negative at every point in the growth region. In this manner, the displacement in the growth region is required to possess a negative gradient (at least in a global sense), while locally maintaining a positive gradient over most of the same region. These two seemingly contradictory requirements are satisfied by the discontinuous displacement field depicted in Figures 3 through 5.

4 Conclusions

A theory of surface growth in continua is proposed that generalizes the notion of a reference configuration, since the body includes material particles that come to existence at different times, due to the process of surface growth. This results in a reference configuration that is (implicitly) a function of time through the time-dependent addition/removal of material particles. One important implication of this generalization is that the displacement (and its gradient) needs to be calculated relative to placements corresponding to different times for different parts of the body.

In this work, an incremental formulation is adopted, whereby a reference configuration is chosen to be sufficiently close in time to the current configuration. Also, a decomposition of the deformation and growth behaviors is effected to enable the analysis of both processes. The balance laws are formulated with respect to a non-inertial frame of reference, which is used to track the apparent motion of the body. A simple one-dimensional problem provides initial evidence of the applicability of the theory. The example illustrates that, for certain sets of boundary conditions, one solution involves a displacement field with layers of discontinuities along surfaces which, at earlier times, were exterior boundaries experiencing surface growth. It is hoped that this theory will contribute to the systematic analysis of two- and three-dimensional models of crawling motile cells and other classes of bodies experiencing surface growth.

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Figure 1: A schematic depiction of typical configurations \mathcal{R}^{τ} (reference), $\tilde{\mathcal{R}}^{\tau+t}$ (intermediate) and $\mathcal{R}^{\tau+t}$ (current) in the theory of surface growth.



Figure 2: Configurations \mathcal{R}^0 , $\tilde{\mathcal{R}}^{t_1}$ and \mathcal{R}^{t_1} of the growing body for $\bar{v}_A = 0$, $\bar{v}_B > 0$, $\bar{v}_{gA} > 0$ and $\bar{v}_{gB} > 0$. Corresponding displacement fields are depicted above each configuration.



Figure 3: Configurations \mathcal{R}^{τ_2} and $\tilde{\mathcal{R}}^{\tau_2+t_2} = \mathcal{R}^{\tau_2+t_2}$ of the growing body for $\bar{u}_A = \bar{u}_B = 0$, $\bar{v}_{gA} = \bar{v}_{gB} = 0$. Corresponding displacement fields are depicted above each configuration.



Figure 4: Displacement at time t = 1.0 for different values of n. The circle in the lower right sub-plot defines the region shown in detail in Figure 5.



Figure 5: Detail of displacement at time t = 1.0 for n = 100.



Figure 6: Displacement at time t = 1.0 for n = 1 and for n = 100.