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On a Patch Test for Contact Problems in Two Dimensions

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Abstract

In this paper we propose a simple finite element completeness check in the spirit of the patch test originating in the work of Irons, with particular reference to frictionless contact problems in two dimensions. Attention is primarily focused on classical slideline treatments based on node-to-node or node-to-surface geometric assumptions. Representative element meshes are subjected to the proposed test via a penalty formulation and conclusions are drawn for both linear and quadratic order elements. In view of encountered shortcomings, a more general slideline methodology is advocated.

1 Introduction

Boundary-value problems involving mechanical contact between deformable bodies are of considerable practical interest in engineering applications. In the context of finite element methods, algorithms for the numerical simulation of contact phenomena originally appeared in the technical literature of the early seventies, [1,2]. In the following years until today the methodologies have substantially improved following closely the growth that finite elements have generally experienced within the realm of structural mechanics.

A pivotal objective in contact numerical algorithms is the accurate transmission of pressure forces from one body onto the other, coupled with observance of non-penetration. This two-fold task is not always an easy one. The geometric constraint of impenetrability is by nature *unilateral*, in the sense that it is enforced only on the parts of the body surfaces where contact actually takes place, while it remains inactive elsewhere. An additional complication stems from the fact that, in most cases, the exact location and extent of the contacting surfaces is not known to the analyst in advance.

Algorithms for finite element simulations of contact problems depend crucially on two tacitly related assumptions. These are the discretization of boundary surfaces potentially in contact and the approximation of pressure distribution within the regions of actual contact. Geometric considerations prevail in identifying the fictitious contact elements, whereas stability and accuracy guide, to a point, the choice of pressure fields.

In this paper we revisit classical slideline treatments based on the node-to-node and node-to-surface assumptions, [3,4], and attempt an assessment of their robustness through the introduction of a simple benchmark problem, referred to as a *patch*

test. Standard arguments show that satisfaction of the patch test constitutes a necessary condition for algorithmic convergence. Failure to conform with its requirements typically results in loss of accuracy.

The constrained extremization problem of unilateral contact can be numerically approximated by means of formulations directly related to the Lagrange multiplier concept. Here, we employ a penalty formulation to free the space of admissible displacements from restrictions pertaining to contact constraints. The behavior of the tested algorithms is evaluated and explained, while shortcomings are identified for both linear and quadratic elements. The conclusions obtained are used to motivate alternative methodologies.

2 Classical slidelines: a brief review

Node-to-node and node-to-surface slideline treatments are used extensively in numerous research and commercial finite element programs that model contact phenomena. Here, the underlying assumption is that contact between interacting bodies will be accounted for with reference to the boundary nodal points only. The NIKE/DYNA codes at the Lawrence Livermore National Laboratory were among the first to adopt the above methodology and test the resulting numerical algorithms in large scale computations, [5].

The node-to-node method presupposes a one-to-one correspondence between boundary nodes on the contacting bodies. This is a geometrically limited assumption in that it can be easily violated, often to unacceptable level, when large body motions take place during the loading process. To its advantages one may recognize the obvious algorithmic simplicity and the unbiased treatment of the contacting surfaces. The assumed pressure field emanating from the node-to-node logic is essentially an equivalent force field concentrated at the nodes that are in contact, see Fig. 2.1.

The node-to-surface method permits a *slave* node to hit at a generic point across the opposite *master* surface in the sense of Fig. 2.2. This logic allows for arbitrarily large motions to be treated accurately, since changes in contact geometry can be directly accounted for by the algorithm at every given load step. It should be noted, however, that this procedure invokes bias between the two bodies as the nodes of the one (the slave) hit the surface of the other (the master). This is the 'one-pass' version of the method. The corresponding pressure field is shown in Fig. 2.3. Clearly, one can eliminate the geometric bias by reversing the role of master and slave surfaces and repeating the same process, thus obtaining the 'two-pass' version illustrated in Fig. 2.4.

In the above slideline treatments the non-penetration condition is enforced at discrete (nodal) points along the contacting surfaces. At these same points, we obtain contact forces, that is nodal forces desirably equivalent to an accurate approximation of the actual pressure field. In practice, tributary area methods are sometimes used to *a posteriori* recover a pressure field from the concentrated forces.

In the following section, we address the accuracy of pressure transmission along the contact surfaces, in the context of the above classical slideline treatments.

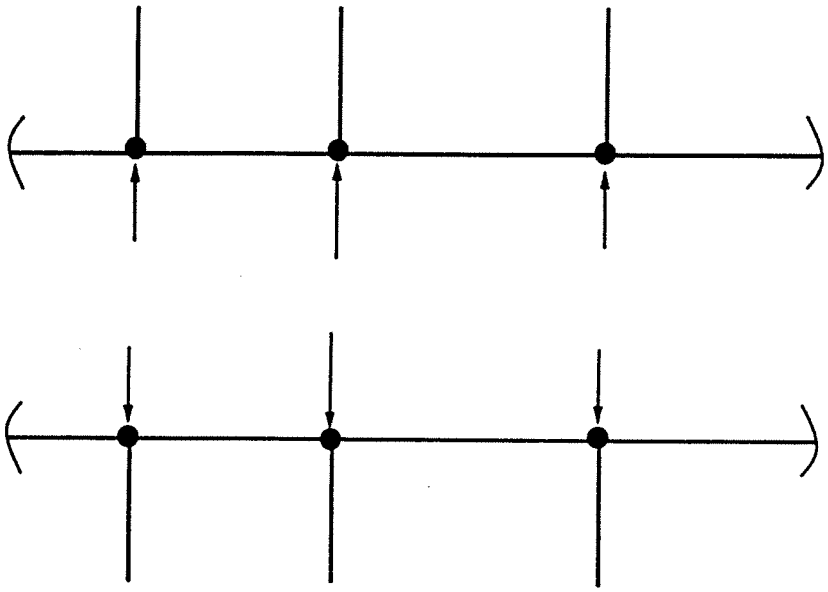

Fig. 2.1 Node-to-node contact

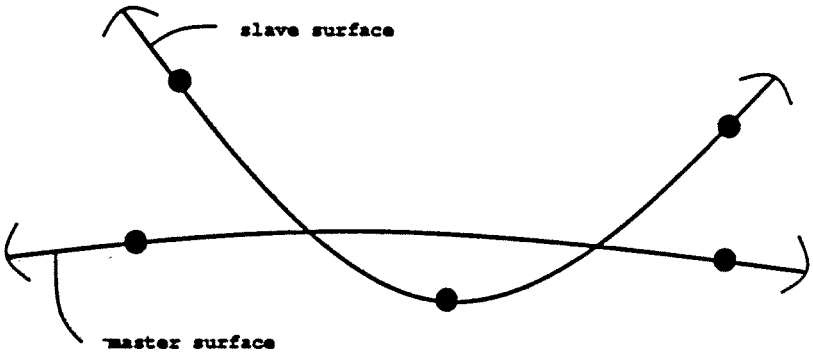

Fig. 2.2 Node-to-surface contact

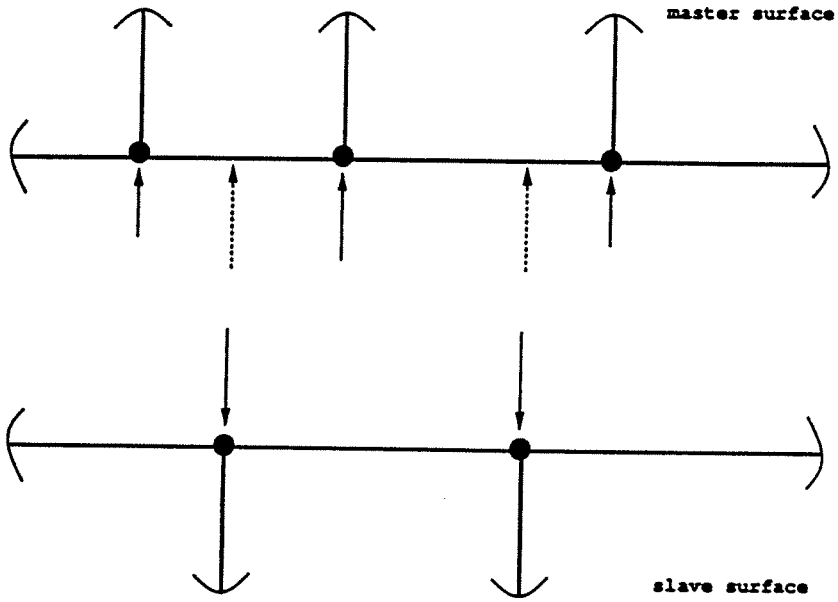

Fig. 2.3 Node-to-surface contact: one-pass algorithm

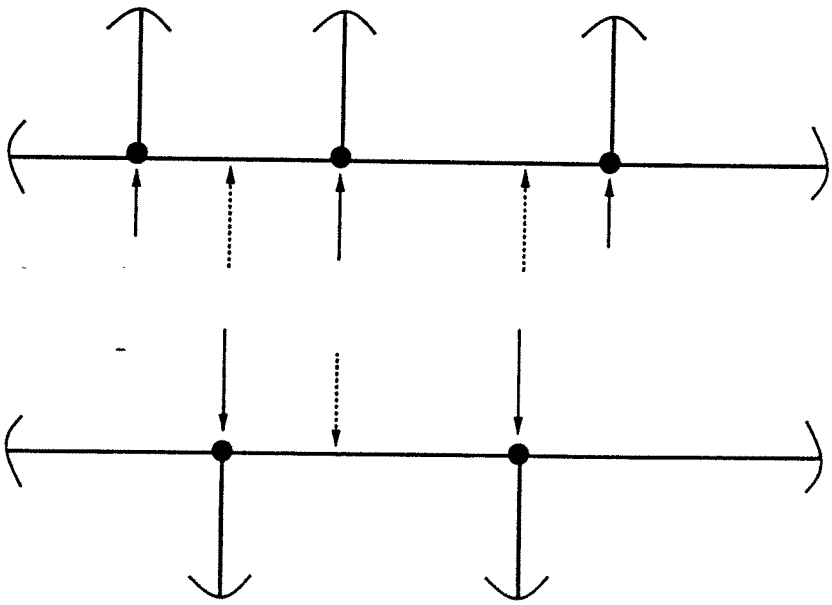

Fig. 2.4 Node-to-surface contact: two-pass algorithm

3 A patch test for contact problems

We propose a simple two-dimensional completeness test for finite element formulations of contact problems. The original idea of Irons, [6], extended to mixed finite elements by Zienkiewicz *et al.*, [7], is used here to assess the robustness of the contact methodology. The role of the patch test in convergence is discussed in detail by Zienkiewicz and Taylor, see [8]. In the spirit of the Irons patch test and its generalizations, the requirement for exact reproduction of a constant strain field gives way to a similar requirement pertaining to the (equivalent) pressure field, recovered by the algorithm along the contact surfaces.

Based on the above rationale, we introduce the problem of Fig. 3.1. A rectangular elastic punch is forced into a finite elastic foundation by means of a uniformly distributed force field. Since both bodies have the same material properties, the exact solution invokes constant pressure throughout the area of contact (this is also true for different material properties in the two bodies if a frictionless interface is assumed). A higher order patch test involving the application of linearly distributed force on top of the same body system may also be conducted. In this paper we restrict ourselves to the constant pressure test.

All computations have been performed within the environment of the fully nonlinear finite element program FEAP, which is partially documented in [8]. Small deformation linear and quadratic isoparametric elements in plane strain have been used throughout the analysis. The constraint conditions are treated by a classical penalty formulation.

The element meshes considered in this work are shown in Fig. 3.2. The following table conveniently summarizes the test results obtained.

Patch test results							
	Element meshes						
Pressure	A	B-1p	B-2p	C-1p	C-2p	D	E
Constant	p	p/f	p	f	p	p	f

p: pass f: fail

The above results are analyzed in the following section.

4 Evaluation of slideline treatments

4.1 Summary of the governing equations

We briefly summarize the equations governing the boundary-value problem of unilateral mechanical contact in the absence of friction, as formulated by means of a classical penalty method. To this end, we define the augmented energy functional Π_ϵ according to

$$\Pi_\epsilon(\mathrm{u}) = \Pi_{TPE} + \int_C \frac{1}{2}\epsilon h^2 \, d\Gamma \,, \tag{4.1}$$

where Π_{TPE} is the usual total potential energy for the bodies and $h(\mathrm{X}, \mathrm{u})$ is the

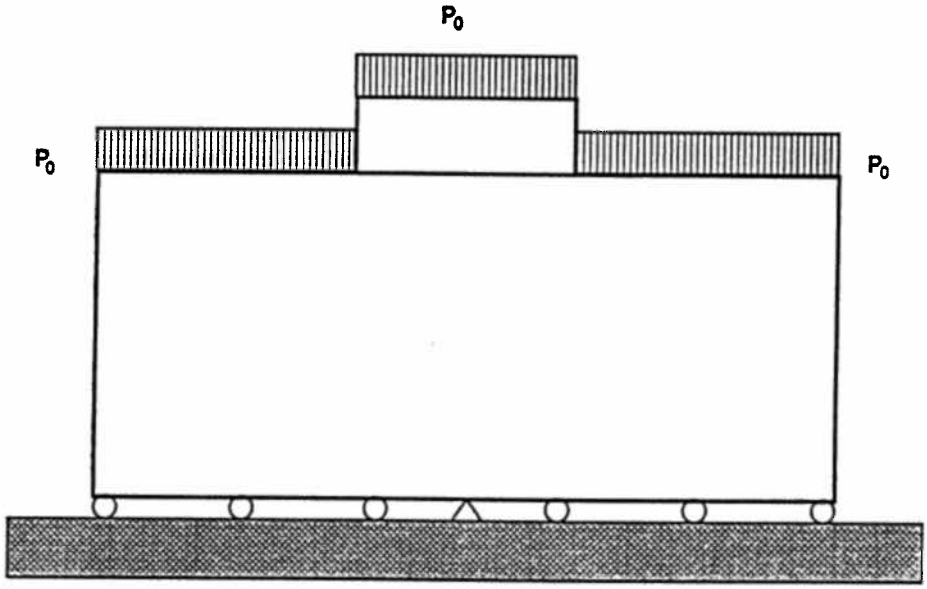

Fig. 3.1 Constant pressure patch test

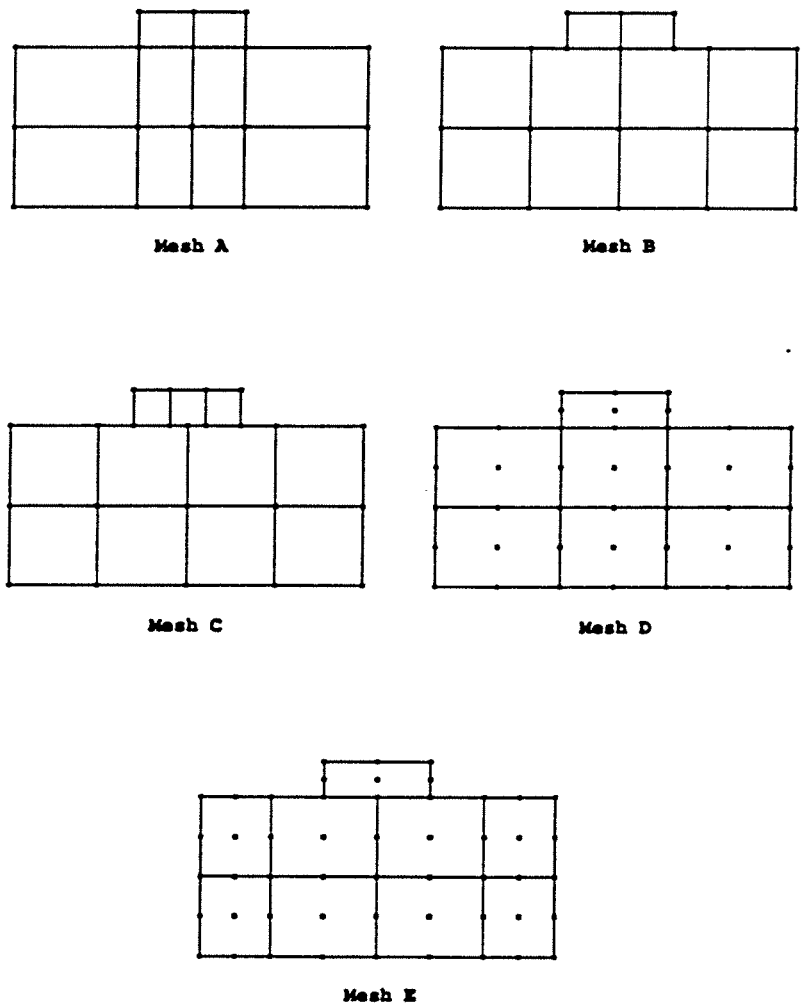

Fig. 3.2 Finite element meshes for contact patch test

local gap (penetration) function along the actual contact surface C, depending upon the boundary displacements of both bodies, $\mathbf{u}$, and the surface coordinates in the reference configuration, $\mathbf{X}$. The geometric constraint of non-penetration requires that

$$h = 0 \tag{4.2}$$

along the contact surface. A penalty parameter $\epsilon > 0$ controls the degree to which the above constraint is imposed. It is important to note that (4.2) becomes inactive if contact does not occur or release (i.e. positive pressure) is detected. Extremization of (4.1) with respect to the unconstrained displacement field V gives rise to the equation

$$\delta\mathbf{u} \cdot \mathbf{R} + \int_C \delta h \, \epsilon h \, d\Gamma = 0 \, , \tag{4.3}$$

where $\mathbf{R}$ represents the residual force emanating from (unconstrained) extremization of the total potential energy functional. Clearly, the higher the value of ϵ, the more accurately (4.2) is enforced. The quantity ϵh can be regarded as the penalty force associated with the approximate satisfaction of (4.2).

A Lagrange multiplier formulation can be realized by introducing the functional

$$\Pi_L(\mathbf{u}, p) = \Pi_{TPE} + \int_C p \, h \, d\Gamma \, , \tag{4.4}$$

where the pressure field P is independently assumed at the outset. Finally, an augmented Lagrangian functional Π_{AL} combining a penalty with a classical Lagrange multiplier method can be defined according to

$$\Pi_{AL}(\mathbf{u}, p) = \Pi_{TPE} + \int_C \frac{1}{2} \epsilon h^2 \, d\Gamma + \int_C p f(h) \, d\Gamma \, ,$$

with $f(0) = 0$ (e.g. $f(h) = h$). Standard arguments show that both penalty and augmented Lagrangian formulations converge to the Lagrange multiplier solution in the continuum setting as the penalty parameter approaches infinity. Thus, in what follows it is justified to consider all three formulations under the same line of thought.

4.2 Pressure distribution

The purpose of this subsection is to propose a simple analysis of the behavior exhibited by the slideline methods under consideration, when subjected to the patch test in conjunction with the family of Lagrange multiplier formulations outlined above. It will be shown that accurate representation of a constant pressure field is attainable by some methods, depending on how integration of the momentum balance equation (4.3) is performed along the contact surface. It should be noted that, given a particular slideline treatment, *a single mesh for which the patch test fails is sufficient to ascertain overall lack of robustness.*

In a discrete setting, the finite element approximation for the displacement field gives rise to an approximation of the local gap (penetration) function in the form

$$h = \mathbf{H}(\mathbf{X}, \mathbf{u}) \, \mathbf{u} + \mathbf{H}_0(\mathbf{X}) \, ,$$

where $\mathbf{H}$, $\mathbf{H}_0$ involve degrees of freedom associated, in principle, with surface nodes. With the additional assumption of constant pressure p_0 distributed throughout the region of contact, C, the penalty term in (4.3) becomes

$$\int_C \delta \mathbf{u}\, p_0\, \mathbf{H}\, d\Gamma \,. \tag{4.5}$$

It is the integration of this particular term that governs the behavior of the various slideline treatments. In view of the above conjecture, let us attempt to explain the patch test results obtained by the meshes of the previous section.

In meshes A though C, where linear elements are used, the integrand in (4.5) is a piecewise linear function in space. Clearly, the node-to-node contact of mesh A allows for nodal quadrature to be equivalent to the trapezoidal rule within each of the two symmetrically formed segments. Therefore, integration of (4.5) is exact and the patch test is passed.

In mesh B we distinguish between one- and two-pass algorithms. Moreover, within the one-pass procedure, there exist two alternatives; indeed, each of the two bodies may serve as the slave allowing the honor of being a master to the other. It is evident that the one-pass algorithm obtained by rendering the foundation master is practically equivalent to the two-pass algorithm. This is because, in this case, reversal of the master-slave characterization will not enhance the force distribution, but it will merely add stiffness to the center node. It follows that both formulations will pass the patch test, since, again,
integration of (4.5) is performed by nodal quadrature within each of the segments on which the integrand in (4.5) is a linear function. The remaining one-pass algorithm unambiguously fails the patch test, because it attempts to integrate a piecewise linear function with only one quadrature point (corresponding to the center node).

Mesh C can be regarded as representative of a wide class of meshes encountered in contact finite element discretizations. Here, the distribution of nodes is arbitrary with respect to the corresponding nodes on the opposite surface. None of the two potential one-pass algorithms is successful. The rationale is the same as before; there exist four segments, in which the integrand of (4.5) is piecewise linear, and, if we consider the punch as slave (resp. master), there are four (resp. one) quadrature points to integrate it with. The two-pass algorithm has exactly as many nodal integration points as needed (i.e. five) for accurate integration of (4.5). Furthermore, elementary geometry dictates that the contact surface obtained in the context of the two-pass algorithm with arbitrary node distributions will be invariably flat, provided that contact conditions are detected for all boundary nodes. This is an undesirable feature whenever the algorithm is used to tie two meshes with arbitrary boundary node distribution, because it disallows accurate modeling of warping or bending along the tied surface.

Meshes D and E are constructed by 9-node quadratic elements, which, when undistorted, render the integrand of (4.5) piecewise cubic in a suitable local coordinate system. The three-point nodal quadrature of mesh D (Simpson's rule) corresponding to the node-to-node contact of three nodes is adequate for the purpose of satisfying the patch test. The same applies even if the contacting elements are distorted, although still flat (note that this case is entirely equivalent to having two

a priori connected flat elements, which, allow, as is well known, exact transmission of constant surface forces).

Mesh E fails the patch test. This can be simply explained by the insufficient number of integration points (i.e. three points) involved in integrating the two piecewise cubic terms pertaining to the given mesh. The same test is similarly failed by one- or two-pass node-to-surface formulations.

Extensive numerical experimentation has confirmed the above findings. In summarizing, we claim that one-pass algorithms for linear elements and all nodal-based algorithms for quadratics fail, in general, the proposed patch test. Exceptions are limited and pertain to special meshes, which are, indeed, of little use whenever large body motions occur. On the other hand, the two-pass algorithm passes the test for all linear element meshes.

4.3 Solvability conditions for the discrete problem

A discrete contact algorithm is subject to solvability conditions, owing to the geometric constraint of impenetrability. The choice of pressure approximation is pivotal in rendering the problem solvable, as well as in providing a stable solution. As it becomes apparent, there are two diametrically opposed, but equally significant conditions to be met. First, the pressure approximation should be adequately low in order, or discretized in such a way that it be 'computable' given the number of available active displacement degrees of freedom, in the spirit of the *constraint counts* in [7]. Second, the same pressure should possess enough approximating power to accurately and stably represent the exact field.

By their construction, contact algorithms based on nodal logic (including both node-to-node and node-to-surface methods) satisfy the constraint counts. This is because, in this case, unknown pressures are always associated with active boundary nodes. Stability is not as easy to assess. Numerical experience with linear elements shows that the two-pass algorithm is generally stable (recall that it involves the maximum allowable number of independent pressure quantities!). One-pass algorithms are potentially unstable depending on the particular topology of the meshes, see for example mesh B of Fig. 3.2.

Keeping in mind the patch test along with the solvability requirements, we proceed in proposing an alternative approach to nodal treatments, which is conceptually more general and applicable to all order elements.

5 Contact segments: a general approach

The fundamental assumption of nodal contact is the discretization of the contact surface into contact elements, each of them consisting of a node on the slave surface and a corresponding node or element edge on the master surface. The resulting pressure field is buried into the nodal forces computed by the algorithm and is only artificially recovered. The computational significance of this approach must be appreciated, especially in light of its straightforward three-dimensional generalization. Here, we advocate an alternative, more general approach based on a continuous discretization of the contact surfaces into *contact segments*, along which a distributed pressure field is assumed. For an earlier attempt in the same direction, see [9].

In order to motivate this approach let us reconsider mesh B of Fig. 3.2. Two contact segments may be easily defined by the end points of the element edges along the contact surface. For each segment we make an approximation for the pressure field. It can be easily inferred that continuous piecewise linear pressure field and integration by trapezoidal rule, recovers the two-pass algorithm. Discontinuous piecewise constant pressure field (essentially equivalent to integrating (4.5) with one point quadrature within each segment) yields an alternative method. In both cases the patch test and the constraint counts are satisfied. However, the reduced order of the pressure field gives rise to stability problems, realized in the form of equi-oscillations in the computed pressure, as illustrated in Fig. 5.1. Note that this behavior is not necessarily suppressed by mesh refinement. Oden, [10], seems to reject all algorithms that employ discontinuous pressure fields for the Signiorini problem (i.e. when the foundation is rigid) and this is in accordance with our experience. It may be concluded that for linear elements the segments provide, on the one hand, a more general algorithmic framework, but, on the other hand, do not offer a more attractive alternative to the two-pass treatment.

For quadratic elements, the segmental logic provides new insight to the problem. Geometrically unbiased segments may now be defined as in Fig. 5.2, so that each one be associated with only a single element edge from each body. By way of continuous pressure approximation and Simpson's integration within a segment, the constant patch test is passed, contrary to the two-pass algorithm, while constraint counts and stability are not sacrificed. Moreover, contact calculations retain their local nature and may be performed within a finite element program employing a nodal logic with minimal additional effort. Details of this methodology may be found in [11].

Conclusion

In this work, robustness of contact algorithms has been addressed. Attention has been focused on pressure approximation and its algorithmic implications, rather than merely on satisfaction of the non-penetration condition. It has been shown that some nodally based algorithms fail to simulate exactly constant pressure conditions, owing to the inadequate approximation of the pressure distribution. Stability considerations are shown to further restrict the choice of pressure fields. Finally, a more general treatment is shown to emanate from continuous discretization of the contact surfaces and the associated pressures.

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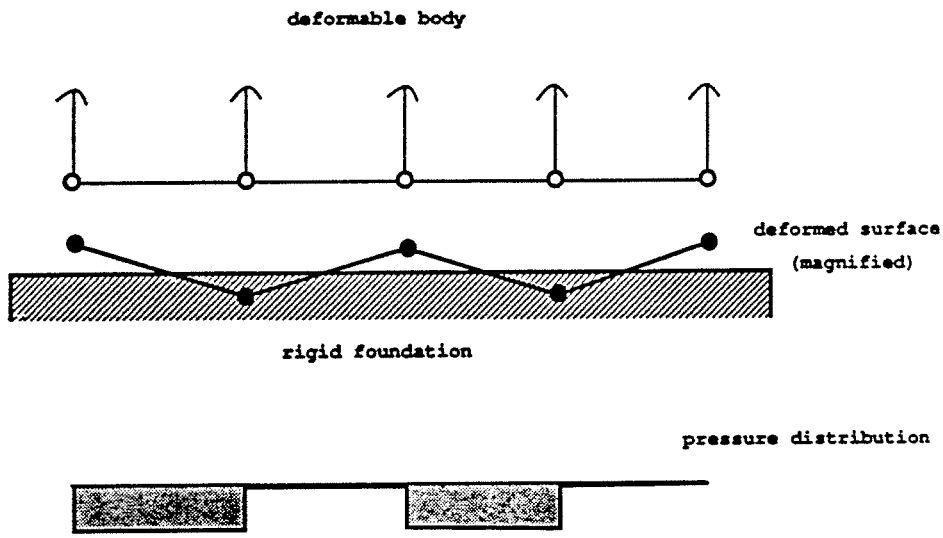

Fig. 5.1 Contact segments for linear elements: pressure oscillation due to discontinuous constant pressure approximation

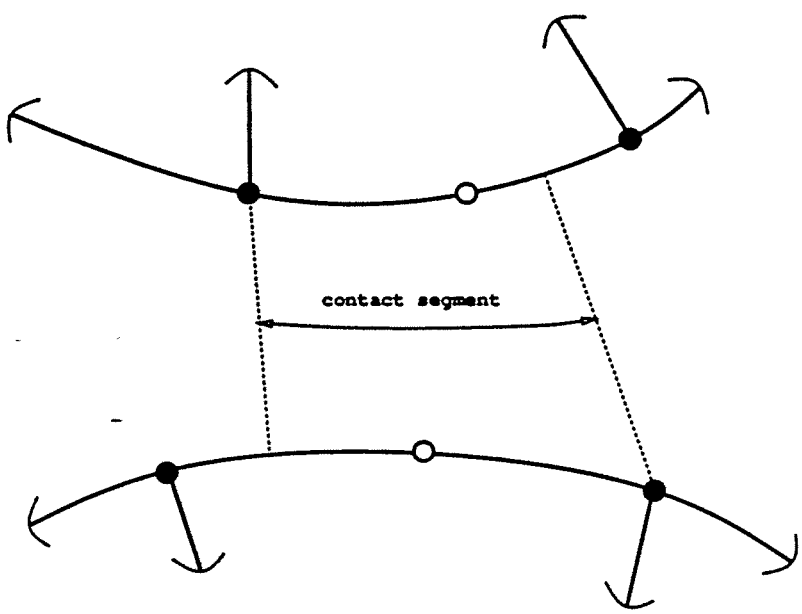

Fig. 5.2 Definition of contact segments for quadratic surfaces

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